2.3

1. Suppose $Q$ is an orthogonal matrix with size $n \times n$, prove that $\det Q$ is either 1 or $-1$.
2. Suppose $A$ is $3 \times 3$ matrix. $A_{ij} = 0$ if $i > j$. Show that $\det A$ is product of the elements on the diagonal using both the definition $\det(A) = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$ and Laplace’s expansion with respect to the first row.

Second Part: Vector Calculus

1.1

1. Let $\vec{r}(t) = [t^2, 3t^3]^T$. Find the arclength $s(t)$ and parametrize this curve using $s$.
2. For $\vec{r}(t)$, we have the arclength $s(t)$. $v = \frac{ds}{dt}$ is the speed and $\vec{v} = \frac{d\vec{r}}{dt}$ is the velocity. Use chain rule to differentiate $\vec{r}(t) = \vec{r}(s(t))$ and prove that $|\frac{d\vec{r}}{ds}| = |\frac{d\vec{v}}{dt}| = 1$.
3. $\vec{r}(t) = [x(t), y(t), z(t)]^T = [t^{\sqrt{73}}, \cos^{100}(2t) - 9\sin(\sqrt{7}t), \tan(5t^{1.2})]^T$. $s(t)$ is the arclength.
   a). Find the integral $\int_2^5 |\frac{d\vec{r}}{ds}| ds$. What’s the length of this curve between $s(t_1) = 2$ and $s(t_2) = 5$?
   b). Find the integral $\int_3^7 \sqrt{(dx/ds)^2 + (dy/ds)^2 + (dz/ds)^2} ds$. What’s the length of this curve between $s = 3$ and $s = 7$?

1.2

#1, #2 #4 #6

More:
   a). Use equation (2) to convince yourself that $\int_C \vec{F} \cdot d\vec{r}$ is the work you learned in physics.
   b). A charged particle with charge $q_1$ is located at the origin. Another particle $B$ with charge $q_2$ is moving. The electric forces between the two particles acting on $B$ is $\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \hat{r}$. Suppose particle B is moving from $2\hat{i}$ to $3\hat{k}$ along one special curve. Find the work the electric force did during this process.
   c). (i). Consider the circle centered at the origin with radius $r$ in 2D plane. Suppose a particle is moving counterclockwises from $A(r, 0)$ along the circle. Parametrize the trajectory of the particle for one rotation ($\vec{r} = x\hat{i} + y\hat{j}$).
     (ii). If there is a force field in (i) which is $\vec{F}(\vec{r}) = F_0 \hat{i}$, calculate the work done on the particle.
     (iii). If the force field in (i) is $\vec{F}(\vec{r}) = \frac{-y}{x^2+y^2} \hat{i} + \frac{x}{x^2+y^2} \hat{j}$, do (ii) again.

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(iv). Also consider the circle. If the line density of the circle is \( \lambda(\vec{r}) = |\vec{r}|^3 \). The total mass of the circle would be \( m = \int_C \lambda(\vec{r})|d\vec{r}| \). Get this total mass.

(v). The circle is charged with charge line density \( \lambda \vec{r} = \lambda_0 \) which is a constant. There is electric field which is tangent to the circle at every point and the strength is \( |\vec{E}| = |x|^2 + |y|^3 \). The direction is counterclockwise. Calculate the total electric force acting on the circle.

(vi). Consider that the circle is a circuit with current \( I \) in flowing counterclockwisely. We have magnetic field \( \vec{B} = \hat{\gamma} \vec{e}_x \). The total force would be \( \vec{F} = \int_C (I d\vec{r}) \times \vec{B} \). Calculate this force.