Part of Hints for to Hw 3

Math 321

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1.12

1. b). $2\hat{i}$ or $-2\hat{i}$ unit is rad/s

2.1

a). $\cos(\pi/3) = 1/2. \cos(\pi/6) = \sqrt{3}/2, 0$ Thus $\vec{e'}_1$ has components $[1/2, \sqrt{3}/2, 0]^T$ with respect to old basis.
b). $[-\sqrt{6}/4, \sqrt{2}/4, \sqrt{2}/2]^T$
c). Use cross product: $[\sqrt{6}/4, -\sqrt{2}/4, \sqrt{2}/2]^T$
d). Rows are just the coordinates we have calculated.
g)(*). The inverse equals the transpose and you can figure out how to solve it.

2.2

#4: $\sum_{i,j} x_i A_{ij} x_j$
#5: Suppose Q_1, Q_2 are orthogonal, then $Q = Q_1 Q_2$. $Q^T Q = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$. Similarly, $QQ^T = I$. Geometrically, the composition of rotations reflections is another rotation plus refection.
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Basically, if we want to find the matrix A for a transformation T(reflection, rotation etc), you can find the coordinates for $Te_1.Te_2, Te_3$ with respect to the basis e_1, e_2, e_3. Denote these coordinates as f_1, f_2, f_3. Then, $A = [f_1, f_2, f_3]$.
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#7:

$$\begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$$

#8

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

#10

$$\begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

#11(*): **Bascially this problem is what #9 requires. However, I don't want to manipulate with Euler Angles. Below, I'll use my own method to solve it.**
Actually, this question is not easy for me neither. Below, I'll derive a general case for you. If you can't understand, it's quite OK.
Suppose we have an orthonormal basis (to be convenient, I won't use arrows here) $\{e_1, e_2, e_3\}$. Suppose Te_i has coordiante y_i which is a column vector. Then, we would have $T[e_1, e_2, e_3] = [e_1, e_2, e_3][y_1, y_2, y_3]$. $A = [y_1, y_2, y_3]$ is a 3×3 matrix.
A is called the matrix representation of T under this basis. For example, $\hat{i}, \hat{j}, \hat{k}$ is the basis. Then, reflection about $x - z$ plane would change them into $\hat{i}, -\hat{j}, \hat{k}$. Then, we get the matrix in #7.
Ok, suppose we have a unit vector $\hat{n}$, which has coordinate $f_3 = [n_1, n_2, n_3]^T$ with respect to e_1, e_2, e_3. We now rotate about $\hat{n}$ right handed by angle α. The question is what is the matrix representation of the transform?
I'll denote this transform as T. To do this, I'll use a new bais e'_1, e'_2, e'_3 where $e'_3 = \hat{n}$. Under this new basis, the matrix representation is quite easy:

$$B = \begin{bmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{bmatrix}$$

Then we actually have:

$$T[e'_1, e'_2, e'_3] = [e'_1, e'_2, e'_3]B = [e'_1, e'_2, e'_3]\begin{bmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{bmatrix}$$

Let's assume e'_1 has coordinate f_1 under the old basis and e'_2 has coordiante f_2 under the old basis. Then we have:

$$[e'_1, e'_2, e'_3] = [e_1, e_2, e_3][f_1, f_2, f_3]$$

We actually have

$$T[e_1, e_2, e_3][f_1, f_2, f_3] = [e_1, e_2, e_3][f_1, f_2, f_3]B$$

However, $Q = [f_1, f_2, f_3]$ is orthogonal. Then we have:

$$T[e_1, e_2, e_3] = [e_1, e_2, e_3]QBQ^T$$

We conclude that this rotation has matrix representation QBQ^T under the old basis.
Recall $Q = [f_1, f_2, f_3]$, we finally derive that

$$A = \cos\alpha(f_1 f_1^T + f_2 f_2^T) + \sin\alpha(f_2 f_1^T - f_1 f_2^T) + f_3 f_3^T$$

f_3 is fixed and we have no choice. f_1, f_2 have one degree of freedom. However, f_1, f_2, f_3 must be orthonormal. But one can prove that the final answer has nothing to do how you choose f_1, f_2. The matrix for $\cos\alpha$ is symmetric and that for $\sin\alpha$ is antisymmetric.

For this problem, $f_3 = [\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3]^T$. We can choose $f_1 = [\sqrt{2}/2, -\sqrt{2}/2, 0]^T$ and f_2 can be obtained by taking the cross product. $f_2 = [\sqrt{6}/6, \sqrt{6}/6, -\sqrt{6}/3]^T$. $f_1 f_1^T + f_2 f_2^T$ becomes:

$$A_1 = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$f_2 f_1^T - f_1 f_2^T$ becomes:

$$A_2 = \begin{bmatrix} 0 & -\sqrt{3}/3 & \sqrt{3}/3 \\ \sqrt{3}/3 & 0 & -\sqrt{3}/3 \\ -\sqrt{3}/3 & \sqrt{3}/3 & 0 \end{bmatrix}$$

$f_3 f_3^T$ becomes:

$$A_3 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Final answer would be $\cos(\alpha)A_1 + \sin(\alpha)A_2 + A_3$