Part of anwsers to Hw 1

Math 321

Without specific declaration, the scalar set is always $\mathbb{R}$, i.e. the set of all real numbers. The problems with $*$ are for the ones who like math.

Section 1.1 #3:

Idea: Denote the set V. For any $f, g \in V$, we need to show that $f + g \in V$. This is used to ensure $f + g$ makes sense. Then check the properties for addition. Similarly, we need to show $\alpha f \in V$ and the properties for scalar multiplication.

Proof: (I'll show the first group of properties as examples)
$\forall f, g \in V$, by definition, the value of $h = f + g$ at $x \in \mathbb{R}$ is $f(x) + g(x)$. Since $f(x), g(x)$ are real, $h(x)$ is real. Then, h is a real function, so $f + g \in V$.
$\forall x \in \mathbb{R}$, $(f + g)(x) = f(x) + g(x)$. $(g + f)(x) = g(x) + f(x)$. Because $f(x), g(x)$ are real numbers, $f(x) + g(x) = g(x) + f(x)$. Since for all x, $(f + g)(x) = (g + f)(x)$, we must have $f + g = g + f$.
$\forall x \in \mathbb{R}, \forall f, g, h \in V$,
$((f + g) + h)(x) = (f(x) + g(x)) + h(x) \overset{*}{=} f(x) + (g(x) + h(x)) = (f + (g + h))(x)$. This is true for any x, we must have $(f + g) + h = f + (g + h)$. Note that $*$ is correct because of the property of real numbers.
$\forall f \in V$, we denote $-f$ as the function whose value at x is $(-f)(x) = -f(x)$. Then, $(f + (-f))(x) = f(x) + (-f(x)) = 0$(zero number). This is true for any x, then $f + (-f) = 0$(zero function, i.e. the function whose value at each x is zero).
$\forall f \in V$, $(f + 0)(x) = f(x) + 0 = f(x) = 0 + f(x) = (0 + f)(x)$. This is true for any x, then we have $f + 0 = f = 0 + f$.
(We now move on to the second group of properties and I'd like to omit here. It's easy but boring. Argument is similar.)

Section 1.2. #7

Soln: We rewrite $\overrightarrow{OB}, \overrightarrow{OC}$ as $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$, $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$.
Plug this back and we have:

$$3\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AC} = 0$$

which is equivalent to $\overrightarrow{AO} = \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC})$.
Assume the midpoint of BC is D, one can easily find that $\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$. This means that A, O, D are on a line. O is on the median corresponding to A.
If we rewrite $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$, and $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$, we get: $\overrightarrow{BO} = \frac{1}{3}(\overrightarrow{BA} + \overrightarrow{BC})$.
Similarly, you just rewrite $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and get $\overrightarrow{CO} = \frac{1}{3}(\overrightarrow{CA} + \overrightarrow{CB})$.
Therefore, O is the intersection of medians.

#10

Ans: Dimension is $2n$. One possible basis(you can have other answers, as long as it contains $2n$ linearly independent vectors.)
$e_1 = (1, 0, \ldots, 0), e_2 = (i, 0, \ldots, 0), \ldots, e_{2n-1} = (0, \ldots, 1), e_{2n} = (0, \ldots, i)$ where $i^2 = -1$.

Section 1.3 #1. Ans: Fh. (Here $F = mg$, $\vec{F} = m\vec{g}$).
#6. Ans: $\vec{v}_{\|} = \frac{\vec{B}\cdot\vec{v}}{|B|^2}\vec{B}$ and $\vec{v}_{\perp} = \vec{v} - \vec{v}_{\|} = \ldots$.
#9: Soln: Using calculus: $|\vec{v}|^2 = (\vec{a} + t\vec{b}) \cdot (\vec{a} + t\vec{b}) = |\vec{a}|^2 + 2t\vec{a} \cdot \vec{b} + |\vec{b}|^2 t^2 = f(t)$. This is a quadratic function. $f'(t) = 2\vec{a} \cdot \vec{b} + 2|\vec{b}|^2 t = 0$. $t = -\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$. You check that $f''(t) > 0$. Then at this point, $f(t)$ achieves its minimum. This vector is $\vec{v} = \vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$. The length is $\sqrt{f(t)} = \sqrt{|\vec{a}|^2 - \frac{(\vec{a}\cdot\vec{b})^2}{|\vec{b}|^2}}$. Geometrically, this makes sense– Pythagoras's theorem $opposite = \sqrt{hypotenuse^2 - adjacent^2}$.

Using geometry: (Draw the figure yourself).
From the figure, you can see that $\vec{v}$ should be perpendicular to $\vec{b}$ when $\vec{v}$ is the shortest. Thus $\vec{v} = \vec{a}_{\perp} = \vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$.
Another way: $\vec{v} \perp \vec{b} \Rightarrow \vec{v} \cdot \vec{b} = 0 \Rightarrow t = -\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$. The remaining part is the same.

Extra exercises

1. What are the dimensions of the following two vector spaces? Why?
   a). $V = \{(0, 0)\}$ as a subspace of $\mathbb{R}^2$.
   b). $V = \{\alpha\vec{a} + \beta\vec{b}, \vec{a} = (1, 2), \vec{b} = (2, 4)\}$.
   Ans: a). Dimension is 0.
   b). Dimension is 1, because $\vec{b}$ and $\vec{a}$ are NOT linearly independent.

2. (*)If the scalars are picked from $\mathbb{C}$ instead of $\mathbb{R}$, what's the dimension of $\mathbb{C}^n$?
   Ans: Dimension now is n.

3. (*)For fun. Consider the set of solutions to the differential equation $y'' - 2y' - 8y = 0$. Is this set a vector space? What's the dimension? Find a basis for it. Find the component of the solution satisfying $y(0) = 0, y'(0) = 1$ with respect to the basis you choose.
   If the equation is $y'' - 2y' - 8y = 1$, answer the questions once again.
   Ans: $V = \{c_1 e^{4x} + c_2 e^{-2x} | c_1, c_2 \in \mathbb{R}\}$. It is a vector space. Dimension is two. Basis $e_1 = e^{4x}, e_2 = e^{-2x}$.
   For the required special solution: we have
   $c_1 + c_2 = 0, 4c_1 - 2c_2 = 1. c_1 = 1/6, c_2 = -1/6$. Thus the component is $(1/6, -1/6)$.
   For the equation $y'' - 2y' - 8y = 1$, the set of solutions isn't a vector space.

4. (*)Think about #11 in exercises 1.2.

Answer is infinity. If you don't like math, **ignore the following**.
Consider $x_k = 1/2^k$.
Define f_n as:
if $x \leq x_n$ or $x \geq x_{n-1}$, $f_n(x) = 0$.
$f_n((x_n + x_{n-1})/2) = 1$.
f_n is linear function on $[x_n, (x_n + x_{n-1})/2]$ and $[(x_n + x_{n-1})/2, x_{n-1}]$. It's easy to see these functions are linearly independent since they are nonzero at different places. We have infinitely many such functions and thus the dimension is infinity.

(The following may be hard for you.) *However, we have different kinds of infinity. One kind is countable infinity and the other kind is uncountable infinity. Countable means they can be ordered like a sequence.*
Then, is the dimension here countable or uncountable?
For infinite-dimension spaces, we have Hamel basis or Shauder basis.
Here, if we don't consider the topology, we are talking about Hamel basis. This is a subset of the set. Any finitely many functions in this subset are linearly independent. Each function in the set should be able to be written as the linear combination of finitely many functions in this subset.
It can be shown that this set we are talking about doesn't have countable Hamel basis since the space is complete even though it's separable (This mean it has a dense countable subset.)
If we talk about Shauder basis (which are defined to be countable), some spaces having uncountable Hamel basis can have Shauder basis. One example is $L^2[0,1]$ which is a Hilbert space. I don't want to explain what Shauder basis is.
Usually, if the space doesn't have a topology, we talk about Hamel basis. However, if it's normed space, we tend to talk about Shauder basis.
Strictly speaking, the dimension should be the number of vectors in Hamel basis. However, people tend to refer to Shauder basis if they just say 'basis' and call the number of Shauder basis the dimension, when coming up with normed vector space.
For this problem, under Hamel sense, its dimension is uncountably infinite