Answers (Hw 14)

Math 321

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3.2

1. #4
   Solution:
   $$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \forall |x| < 1$$
   Replace x with $-x^2$, and we get:
   $$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty}(-x^2)^n = \sum_{n=0}^{+\infty}(-1)^n x^{2n}$$
   To find the radius of convergence, we apply Ratio Test here:
   $$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |-x^2| = x^2$$
   We require $|ratio| < 1$ and thus $|x| < 1$, which implies the radius is 1.

   To understand this, we consider $f(z) = \frac{1}{z^2+1} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$. You can see that it has singularities at $\pm i$. The series fails at these sigularity points. The radius must be less than or equal to the distance between the center and the closest sigularity. In our case, it's just equal to this distance, which is true in most cases.

   #7: Solution: Apply the generalized Cauchy's formula:
   $$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i f^{(n)}(a)}{n!}$$
   if a is inside the contour here.

   In our case, you need to set $a = 0$ and replace n with $n-1$.

   The first integral is thus:
   $$I_1 = \frac{2\pi i}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \cos z|_{z=0}$$

The second integral is

$$I_2 = \frac{2\pi i}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \sin z|_{z=0}$$

Apply the rule $\frac{d\cos z}{dz} = -\sin z$ and $\frac{d\sin z}{dz} = \cos z$:

$$I_1 = \begin{cases} 0 & n \ even \\ \frac{2\pi i}{(n-1)!} & n = 4k+1 \\ -\frac{2\pi i}{(n-1)!} & n = 4k+3 \end{cases}$$

and

$$I_2 = \begin{cases} 0 & n \ odd \\ \frac{2\pi i}{(n-1)!} & n = 4k+2 \\ -\frac{2\pi i}{(n-1)!} & n = 4k \end{cases}$$

2. Calculate $\int_{|z|=4} \frac{1}{(z+1)(z-1)(z+2i)} dz$ without calculating the partial fraction expressions.
   Solution:
   To find the singularities, we can set $(z+1)(z-1)(z+2i) = 0$. The singularities are $-1, 1, -2i$ which are all inside our contour.

   Denote the integral as I. We then deform our contour to the three small circles around these singular points ($C_1 : |z+1| = \epsilon_1, C_2 : |z-1| = \epsilon_2, C_3 : |z+2i| = \epsilon_3$) and we have:

$$\begin{aligned}
I &= \int_{|z|=4} \frac{1}{(z+1)(z-1)(z+2i)} dz \\
&= \int_{C_1} \frac{1}{(z+1)(z-1)(z+2i)} dz + \int_{C_2} \frac{1}{(z+1)(z-1)(z+2i)} dz \\
&+ \int_{C_3} \frac{1}{(z+1)(z-1)(z+2i)} dz \\
&= \int_{C_1} \frac{1/(z-1)(z+2i)}{z+1} dz + \int_{C_2} \frac{1/(z+1)(z+2i)}{z-1} dz + \int_{C_3} \frac{1/(z+1)(z-1)}{z+2i} dz \\
&= 2\pi i \left(\frac{1}{-2(-1+2i)} + \frac{1}{2(1+2i)} + \frac{1}{(-2i+1)(-2i-1)} \right) \\
&= \frac{\pi(-2+i)}{5} + \frac{\pi(2+i)}{5} - \frac{2\pi i}{5} \\
&= 0
\end{aligned}$$

   by Cauchy's formula.

   This is not surprising, because all simple poles are inside our contour. (Sometimes, the infinity is also a singularity and in those cases, the integral is not zero)

3. Calculate $\int_{|z|=2} \frac{1}{z(z+3)^2} dz$ and $\int_{|z|=2} \frac{1}{z^2(z+3)} dz$

Solution: Denote the first as I_1 and the second as I_2. By Cauchy's formula:

$$I_1 = \int_{|z|=2} \frac{1}{z(z+3)^2} dz = \int_{|z|=2} \frac{1/(z+3)^2}{z} dz = 2\pi i \frac{1}{3^2}$$

because the function $f(z) = \frac{1}{(z+3)^2}$ is quite good inside our contour.

$$\begin{aligned} I_2 &= \int_{|z|=2} \frac{1}{z^2(z+3)} dz = \int_{|z|=2} \frac{1/(z+3)}{z^2} dz = 2\pi i \frac{1}{1!} \frac{d}{dz} \frac{1}{z+3} |_{z=0} \\ &= -\frac{2\pi i}{9} \end{aligned}$$

Also this is true because the numerator is good inside our contour.

4. Calculate $\int_{|z|=3} \frac{e^z}{(z-2)^3} dz$
   Solution: Apply Cauchy's formula directly:

$$2\pi i \frac{1}{2!} \frac{d^2}{dz^2} e^z |_{z=2} = e^2 \pi i$$

5. Calculate $\int_{|z|=2} \frac{\sin z}{z^2+1} dz$
   Set $z^2 + 1 = 0$, and we get $z = \pm i$. We get two small circles: $C_1 : |z - i| = \epsilon_1$ and $C_2 : |z + i| = \epsilon_2$

$$\begin{aligned} I &= \int_{|z|=2} \frac{\sin z}{z^2+1} dz \\ &= \int_{C_1} \frac{\sin z}{z^2+1} dz + \int_{C_2} \frac{\sin z}{z^2+1} dz \\ &= \int_{C_1} \frac{\sin z/(z+i)}{z-i} dz + \int_{C_2} \frac{\sin z/(z-i)}{z+i} dz \\ &= 2\pi i \frac{\sin i}{2i} + 2\pi i \frac{\sin(-i)}{-2i} \\ &= \pi \sin i - \pi \sin(-i) = 2\pi \sin i \end{aligned}$$

6. (*)(Challenging problems)
   (This one could be really hard for you)Calculate $\int_{|z|=1/2} \frac{e^z}{\sin(2z)} dz$
   Solution:
   Let's look at the singular points. We set $\sin(2z) = 0$. First of all, we should know where $\sin w = 0$ in complex plane. $\sin w = \frac{e^{iw} - e^{-iw}}{2i} = 0$ and we get $e^{i2w} = 1$. Let $w = x + iy$ and we have $e^{i2x} e^{-2y} = 1$. We take magnitue and we can see $e^{-2y} = 1$ we must have $y = 0$ since y is real. Then, we have $e^{2ix} = 1$ for real x. This means

$2x = 2k\pi$ and $x = k\pi$. We conclude that $\sin w = 0$ only if $w = k\pi$ even if w can be complex. This means all of these zeros are real.
Then $2z = k\pi$ and $z = k\pi/2$.

Inside our contour, we only have $z = 0$ this singularity.
Around this point, we can see $\sin(2z) = 2z - (2z)^3/3! + \ldots$ and we can see, it's a simple pole, since we can have $z(2 - 8z^2/6 + \ldots)$ where $2 - 8z^2/6 + \ldots$ is really a good function.

Then, we have the integrand as $\frac{g(z)}{z}$ where $g(z) = \frac{e^z z}{\sin(2z)} = \frac{e^z}{2-8z^2/6+\ldots}$. By Cauchy's formula, we have:

$$2\pi i g(0) = 2\pi i \frac{e^0}{2 - 0 + \ldots} = \pi i$$

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1. Calculate $\int_0^{2\pi} \frac{1}{2-\sin\theta} d\theta$(Hint: On unit circle, $\sin\theta = \frac{z-z^{-1}}{2i}$)
   Solution: On unit circle, $z = e^{i\theta}, 0 \le z < 2\pi$
   Then, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ by Euler's identity.
   $dz = ie^{i\theta} d\theta$ and thus $d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$. This integral becomes:

   $$\int_C \frac{1}{2 - (z - z^{-1})/(2i)} \frac{dz}{iz} = -2 \int_C \frac{1}{z^2 - 4iz - 1} dz$$

   We want $z^2 - 4iz - 1 = 0$ to find the singularities. To do this, we use our quardratic formula:

   $$z_{1,2} = \frac{4i \pm \sqrt{(-4i)^2 - 4(-1)}}{2 * 1} = \frac{4i \pm \sqrt{-12}}{2} = (2 \pm \sqrt{3})i$$

   The denominator can be factored as $(z - (2+\sqrt{3})i)(z - (2-\sqrt{3})i)$. Only one root is inside the circle that is $(2-\sqrt{3})i$. By Cauchy's formula:

   $$-2(2\pi i)\frac{1}{(2-\sqrt{3})i - (2+\sqrt{3})i} = 2\pi/\sqrt{3}$$

2. #1(Just consider the case $a > b > 0$)
   Solution: Method is the same.

   $$I = \int_C \frac{1}{a + b(z - z^{-1})/(2i)} \frac{dz}{iz} = 2 \int_C \frac{1}{bz^2 + 2aiz - b} dz$$

   The two roots:

   $$\begin{aligned} \frac{-ai \pm \sqrt{-a^2 + b^2}}{b} &= \frac{b}{ai \pm \sqrt{-a^2+b^2}} \\ &= \frac{-bi}{a \pm \sqrt{a^2 - b^2}} \end{aligned}$$

The magnitudes of them are $\frac{b}{a+\sqrt{a^2-b^2}} < \frac{b}{a} < 1$ and $\frac{b}{a-\sqrt{a^2-b^2}} = \frac{a+\sqrt{a^2-b^2}}{b} > \frac{a}{b} > 1$ respectively.

So, there is only one root inside our contour and by Caucy's formula, we have:

$$\begin{aligned} 2 * 2\pi i \frac{1}{z_1 - z_2} &= 2 * 2\pi i \frac{1}{-bi/(a+\sqrt{a^2-b^2}) + bi/(a-\sqrt{a^2-b^2})} \\ &= 2 * 2\pi i \frac{1}{2i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \end{aligned}$$

3. Redo the integral $\int_{-\infty}^{+\infty} \frac{1}{x^4+1} dx$ to make sure you understand.
   Solution: Please refer to you notes.(I just hope you could really understand.)

4. Calculate $\int_0^{+\infty} \frac{\cos x}{x^2+1} dx$.
   Solution: Denote $I = \int_C \frac{e^{iz}}{z^2+1} dz$.
   To find the singularity, we set $z^2 + 1 = 0$ and get $z = \pm i$. If we pick the contour as the $C_1 \cup C_2$ where $C_1 = [-R, R]$ and C_2 is the upper half circle with radius R.
   We can see we only have one singularity i inside the contour provided $R > 1$. Since we'll let $R \to +\infty$, we really only need to consider this case.
   The pole is simple. By Cauchy's formula:

$$I = 2\pi i \frac{e^{i*i}}{i+i} = \pi e^{-1}$$

We then prove that the integral on the semi-circle goes to zero.

$$\begin{aligned} \left|\int_{C_2} \frac{e^{iz}}{z^2+1} dz\right| &= \left|\int_0^\pi \frac{(e^{-R\sin\theta} e^{iR\cos\theta})}{R^2 e^{i2\theta} + 1} iRe^{i\theta} d\theta\right| \\ &\le \int_0^\pi \frac{Re^{-R\sin\theta}}{R^2-1} d\theta \\ &\le \int_0^\pi \frac{R}{R^2-1} d\theta \to 0 \end{aligned}$$

if we parametrize the upper half circle with $z = Re^{i\theta}$ $0 \le \theta < \pi$.
We thus have:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1} dx &= \lim_{R\to\infty} \int_{C_1} \frac{e^{iz}}{z^2+1} dz = \lim_{R\to\infty} \int_{C_1+C_2} \frac{e^{iz}}{z^2+1} dz \\ &= I = \pi e^{-1} \end{aligned}$$

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1} dx = Re(\pi e^{-1}) = \pi e^{-1}$$

The original integral is $\pi e^{-1}/2$