Some Hints for Hw 14

Math 321

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1. #4 The series is $\sum_{n=0}^{\infty}(-1)^n z^{2n}$. You can see that it has singularities at $\pm i$. The distance between the center and $\pm i$ is just the radius of convergence.
   #7: use the Taylor series or the generalized Cauchy's formula. $\frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\cos z|_{z=0}$ and similar thing for the second one. The derivative in the first one is 0 when n even and ± 1 otherwise.

2. Calculate $\int_{|z|=4}\frac{1}{(z+1)(z-1)(z+2i)}dz$ without calculating the partial fraction expressions.
   Ans: $2\pi i(\frac{1}{-2(-1+2i)}+\frac{1}{2(1+2i)}+\frac{1}{(-2i+1)(-2i-1)})$ and you should simplify

3. Calculate $\int_{|z|=2}\frac{1}{z(z+3)^2}dz$ and $\int_{|z|=2}\frac{1}{z^2(z+3)}dz$
   Ans: The first one is $2\pi i/9$ and the second one is $-2\pi i/9$

4. Calculate $\int_{|z|=3}\frac{e^z}{(z-2)^3}dz$
   Ans: $\pi i e^2$

5. Calculate $\int_{|z|=2}\frac{\sin z}{z^2+1}dz$
   Ans: $2\pi\sin i$

6. (*)(Challenging problems)
   (This one could be really hard for you)Calculate $\int_{|z|=1/2}\frac{e^z}{\sin(2z)}dz$
   Ans: Hint: $\frac{g(z)}{z}$. $g(z)=\frac{e^z z}{\sin(2z)}$ is analytical within the contour($\sin(2z)=0$ only at 0 inside this small circle, and other zeros are outside of this circel.) $g(0)=1/2$. Ans is πi

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1. Calculate $\int_0^{2\pi} \frac{1}{2-\sin\theta} d\theta$(Hint: On unit circle, $\sin\theta = \frac{z-z^{-1}}{2i}$)
   Ans: On unit circle, this integral becomes: $\int_C \frac{1}{2-(z-z^{-1})/(2i)} \frac{dz}{iz} = -2\int_C \frac{1}{z^2-4iz-1} dz$.
   The denominator can be factored as $(z-(2+\sqrt{3})i)(z-(2-\sqrt{3})i)$. Only one root is inside the circle. The anser is $-2(2\pi i)\frac{1}{(2-\sqrt{3})i-(2+\sqrt{3})i} = 2\pi/\sqrt{3}$

2. #1(Just consider the case $a > b > 0$)
   I guess the answer is $2\pi/\sqrt{a^2-b^2}$. Method is the same.

3. Redo the integral $\int_{-\infty}^{+\infty} \frac{1}{x^4+1} dx$ to make sure you understand.

4. Calculate $\int_0^{+\infty} \frac{\cos x}{x^2+1} dx$.
   Ans: Calculate $\int_C \frac{e^{iz}}{z^2+1} dz$. We can see we only have one singularity inside the contour. The pole is simple. We get the answer $2\pi i \frac{e^{i*i}}{i+i} = \pi e^{-1}$ for this complex integral.
   We then prove that the integral on the semi-circle goes to zero.

$$
\begin{aligned}
|\int_{C_2} \frac{e^{iz}}{z^2+1} dz| &= |\int_0^{\pi} \frac{(e^{-R\sin\theta} e^{iR\cos\theta})}{R^2 e^{i2\theta}+1} iRe^{i\theta} d\theta| \\
&\leq \int_0^{\pi} \frac{Re^{-R\sin\theta}}{R^2-1} d\theta \\
&\leq \int_0^{\pi} \frac{R}{R^2-1} d\theta \to 0
\end{aligned}
$$

   We thus have:

$$
\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1} dx = Re(\pi e^{-1}) = \pi e^{-1}
$$

   The original integral is $\pi e^{-1}/2$