

## Hw 11

### Math 321

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### One important concept problem

Consider 2D plane. We have  $\vec{r} = x\vec{e}_x + y\vec{e}_y$ .  $\vec{e}_x$  and  $\vec{e}_y$  are constant vectors and thus  $d\vec{r} = dx\vec{e}_x + dy\vec{e}_y$

a). Show that if  $x, y$  are both function of  $t$ , above expression can be reduced to  $(x'(t)\vec{e}_x + y'(t)\vec{e}_y)dt$ . In particular, if  $t$  is picked as  $y$ , then  $x$  is a function of  $y$  and this expression is equivalent to  $d\vec{r} = (x'(y)\vec{e}_x + \vec{e}_y)dy$ .

b). Show that in all cases in a),  $dx = d\vec{r} \cdot \vec{e}_x$  and get the concrete expressions for  $dx$  in these cases. Do the same thing for  $dy$ .

c). If we define  $dr = |d\vec{r}|$  (in previous sections we used  $ds$ ), then  $dr = \sqrt{(dx)^2 + (dy)^2} (= \sqrt{1 + (y'(x))^2}dx)$ . We could have  $d\vec{r} = dr\hat{t}$  where  $\hat{t}$  is the unit tangent vector. Let  $\hat{n}$  be the unit normal vector and  $\hat{n}, \hat{t}$  is right handed. Show that  $dx = \vec{e}_x \cdot \hat{t}dr = -\vec{e}_y \cdot \hat{n}dr$ . Do similar things for  $dy$ . Use these relationships to rewrite equation (84) and get (106)

### 3.4

1.  $A(2, 1)$ ,  $B(3, 2)$  and  $C(5, 10)$ . Let  $A$  be the triangle determined by these three points and  $C$  be the boundary of this region which is oriented counterclockwisely. Use Green's formula and the geometric meaning of  $\int dx dy$  to get the integral

$$\int_C (y dx - x dy)$$

2.(\*) Use Stokes' Theorem to get the integral form of Faraday's law of induction "The induced electromotive force (EMF) in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit" by using the differential form  $-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}$   
Hint: The concrete formula for Faraday's law reads:

$$\int_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

### 3.5

Would equation (105) be correct if  $\hat{s}$  depends on  $x, y$ ? If not, what's the correct formula? (Hint: Use equation (106) to argue. I guess you want to use  $\vec{v} = F\hat{s}$ )  
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### 3.6

By Gauss's Theorem, we would have:

$$\int_V \nabla \cdot \vec{v} dV = \int_S \vec{v} \cdot d\vec{S}$$

Now, we are interested in the case  $\vec{v} = \frac{\hat{r}}{r^2}$  and  $V$  is the ball  $x^2 + y^2 + z^2 \leq R^2$ .

a). Show that  $d\vec{S} = dS\hat{r}$  and thus right hand side is  $\int_S \frac{1}{r^2} dS = \frac{1}{R^2} 4\pi R^2$

b). Show that  $\nabla \cdot \frac{\hat{r}}{r^2} = 0$  when  $r \neq 0$

c). If you plug your answer to b) into left hand side, what would you get? Does this conflict with Gauss's Theorem? Why?

More problems for 3.5 in next set of homework problems.