Part of Hints for Hw 10

Math 321

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Some easy problems

1. c). Here, I just calculate the Jacobian:

$$
\begin{aligned}
J = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{vmatrix} = r^2\sin\theta
\end{aligned}
$$

2. b). $\nabla f = (2, 6, -8)$
c). $(\vec{r} - (1,1,1)) \cdot \nabla f = 0$
d). $\nabla f \cdot (2,1,1)/\sqrt{6} = ?$

3. a). $\vec{v} = \vec{\nabla} f = \frac{1}{r^2}\nabla r = \frac{\vec{r}}{r^3} = ?$
b). 0 for $r \neq 0$

1.7.2

#1: Suppose the region is D. Then the area is $\int_D dxdy$. However, it's hard to evaluate directly. We use new variables $u = xy, v = y/x$. We can caluclate the area in the first quadrant and times 2. For the region in the first quadrant, $x = \sqrt{u/v}$ and $y = \sqrt{uv}$. Then, the area becomes $2\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2} |\frac{\partial(x,y)}{\partial(u,v)}| dudv$
#5: If we use the polar coordinate, we know $\frac{\partial(x,y)}{\partial(r,\theta)} = r$. Then, the original integral can be reduced to $\int_\theta\int_r e^{-r^2}|Jacobian|drd\theta = \int_0^{2\pi}\int_0^\infty e^{-r^2} rdrd\theta = \pi$
Thus, we get $\int_x\int_y e^{-x^2-y^2}dxdy = \pi$. However, the left hand side equals $(\int e^{-x^2}dx)^2$ so, $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$

2.4

#5. I should give an answer to you for this problem. But, I can't get the answer directly by looking at this problem. Maybe, I'll calculate this later and write it here.
$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

#9. I don't know which way is 'quick'. I'll do like this: $\vec{B} = \frac{(-y,x,0)}{x^2+y^2}$. Then,
$div\vec{B} = grad(\frac{1}{x^2+y^2}) \cdot (-y,x,0) + \frac{div(-y,x,0)}{x^2+y^2} = 0$
$curl\vec{B} = grad(\frac{1}{x^2+y^2}) \times (-y,x,0) + \frac{curl(-y,x,0)}{x^2+y^2} = -2\frac{1}{\rho^4}\vec{\rho} \times (-y,x,0) + \frac{2\vec{e}_z}{\rho^2} = -2\frac{\rho^2 e_z}{\rho^4} + \frac{2\vec{e}_z}{\rho^2} = 0.$
Where $\vec{\rho} = (x,y,0)$. Both identities hold for $x^2+y^2 \neq 0$.

Notes:In our model here, we have a line current along z-axis. Then, we know the magnetic field and electirc field should be constant. There's no current density outside z and that's why we get 0 for both (this can be determined by Gauss's law for magnetism and Ampere's circuital law). As what happened at z, it's impossible to explain clearly to you.