

Hw 10

Math 321

Some easy problems

1. The spherical coordinate is (r, θ, φ) . $\vec{r} = r \sin \theta \cos \varphi \vec{e}_x + r \sin \theta \sin \varphi \vec{e}_y + r \cos \theta \vec{e}_z$.
 - a). Calculate dV using $(\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}) \cdot \frac{\partial \vec{r}}{\partial \varphi}$
 - b). Check that the coordinates are orthogonal coordinates and calculate dV using h_1, h_2, h_3
 - c). The volume element for x, y, z is $dx dy dz$. Then, we know dV under the spherical coordinate should be $Jacobian * dr d\theta d\varphi$ where $Jacobian$ is from x, y, z to r, θ, φ . Using this method, get the volume element once again.
2. Consider $f(x, y, z) = 2x + 3y^2 - 4z^2$.
 - a). Verify that $(1, 1, 1)$ is on the isosurface $f(x, y, z) = 1$.
 - b). Calculate $\vec{\nabla} f$ at $(1, 1, 1)$
 - c). Calculate the tangent plane for the isosurface at the point $(1, 1, 1)$.
 - d). Calculate $\frac{\partial f}{\partial n}$ along \hat{n} which is parallel to $(2, 1, 1)$
3. Consider the function $f(x, y, z) = -\frac{1}{r} = -\frac{1}{\sqrt{x^2 + y^2 + z^2}}$
 - a). Calculate $\vec{v} = \vec{\nabla} f$ for $r \neq 0$
 - b). Calculate $\vec{\nabla} \cdot \vec{v}$ for $r \neq 0$

1.7.2

#1, #5

2.4

#1, #4, #5, #6 #7 #8 #9. More 1: verify (66), (67)

More 2: I know #4 and #5 need some calculation. Let's now digest them using the properties of vectors. To do this, let's begin with $\nabla \times (\nabla \times \vec{v})$. ∇ is a vector operator. For just vectors (no action on whatever follows it), we have $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$. Basically, we can apply this rule to this operator expression, but we need to take into consideration that ∇ has actions on whatever follows it. Here, we can see both ∇ should act on \vec{v} . If we apply our identity here, we would get $\nabla \times (\nabla \times \vec{v}) = (\vec{\nabla} \cdot \vec{v})\vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla})\vec{v}$. However, we have trouble with the first term, since the ∇ wants to act on \vec{v} . Then, we

should change the order and get $\nabla \times (\nabla \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{v}$. Let's move to (70).

We can see that ∇ should act on both \vec{v} and \vec{w} by product rule of differentiation. Then, by product rule we can get the following intermediate thing (illegal to write out in your formal solutions, but reasonable to understand):

$$\nabla \times (\vec{v} \times \vec{w}) = \nabla_v \times (\vec{v} \times \vec{w}) + \nabla_w \times (\vec{v} \times \vec{w})$$

The subindex means which object it acts on. Then, apply our identity for vectors, we would get:

$$\begin{aligned} \nabla \times (\vec{v} \times \vec{w}) &= \nabla_v \times (\vec{v} \times \vec{w}) + \nabla_w \times (\vec{v} \times \vec{w}) \\ &= (\nabla_v \cdot \vec{w})\vec{v} - (\nabla_v \cdot \vec{v})\vec{w} + (\nabla_w \cdot \vec{w})\vec{v} - (\nabla_w \cdot \vec{v})\vec{w} \end{aligned}$$

Now, we want to change back to ∇ . We know this operator acts on what follows it, then we should adjust above to:

$$\begin{aligned} \nabla \times (\vec{v} \times \vec{w}) &= \nabla_v \times (\vec{v} \times \vec{w}) + \nabla_w \times (\vec{v} \times \vec{w}) \\ &= (\nabla_v \cdot \vec{w})\vec{v} - (\nabla_v \cdot \vec{v})\vec{w} + (\nabla_w \cdot \vec{w})\vec{v} - (\nabla_w \cdot \vec{v})\vec{w} \\ &= (\vec{w} \cdot \nabla)\vec{v} - (\nabla \cdot \vec{v})\vec{w} + (\nabla \cdot \vec{w})\vec{v} - (\vec{v} \cdot \nabla)\vec{w} \end{aligned}$$

Now, your task is to understand (71)