Solutions to Quiz 2

- Consider $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$. Our goal is to write it in terms of dot products only.

  1). Use vector identity to do this. (4')

  Soln: Applying the property $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$ here, we know:

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) &= \vec{a} \cdot (\vec{b} \times (\vec{a} \times \vec{b})) \\ &= \vec{a} \cdot ((\vec{b} \cdot \vec{b})\vec{a} - (\vec{b} \cdot \vec{a})\vec{b}) \\ &= (\vec{b} \cdot \vec{b})(\vec{a} \cdot \vec{a}) - (\vec{b} \cdot \vec{a})(\vec{a} \cdot \vec{b})\end{aligned}$$

  2). Use index notation. (6')

  Ans: Using the fact $\vec{a} \times \vec{b} = \epsilon_{ijk} a_i b_j \vec{e}_k$, we have the original expression equal to:

$$(\epsilon_{ijk} a_i b_j \vec{e}_k) \cdot (\epsilon_{lmn} a_l b_m \vec{e}_n) = \epsilon_{ijk} a_i b_j \epsilon_{lmn} a_l b_m \delta_{nk} = \epsilon_{ijk} \epsilon_{lmk} a_i b_j a_l b_m$$
$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_i b_j a_l b_m = a_i b_j a_i b_j - a_i b_j a_j b_i = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$

- (Bonus) Show that $\epsilon_{jki} a_i a_k$ is 0. (3')

  Ans: Method 1:

$$\begin{aligned}&\epsilon_{jki} a_i a_k = \epsilon_{kij} a_i a_k = \epsilon_{kij} a_k a_i \\ &= (\vec{a} \times \vec{a})_j = (\vec{a} \times \vec{a}) \cdot \vec{e}_j = 0\end{aligned}$$

  Method 2:

$$\begin{aligned}&\epsilon_{jki} a_i a_k = \epsilon_{jik} a_k a_i \qquad \textit{changing dummies} \\ &= -\epsilon_{jki} a_k a_i \quad \textit{property of } \epsilon\end{aligned}$$

  This means what we want to calculate equals -1 times itself. This number must be 0