Summary

In this summary, I usually use x as the variable(instead of t). In the spring-mass system, I may use x for the function

1 Previous knowledge

1. Set notations($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \in, \subset, \notin$ etc), set builder notation; Function notations(arrow notation); Integral formulas, integration by parts, trig substitution etc; fundamental theorem, Mean value theorem. (For this part, just need to know them)

2. Techniques for first order equations

   Terminology: Order means the order of highest derivative

   - First order linear equation

     $$y'(x) + p(x)y = q(x)$$

     Notice that this equation may have non-constant coefficients(e.g. $y' + xy = \sin x$ is linear but coefficients are non-constant.) The technique in previous course is integrating factor:

     $$\rho(x) = e^{\int p(x)dx}$$

     Multiplying the factor, one obtains

     $$\rho(x)(y' + p(x)y) = (\rho(x)y)' = \rho(x)q(x)$$

     Integrating yields the GENERAL solution $y = \frac{1}{\rho(x)} \int \rho(x)q(x)dx$. **One can also use the theory below to find the general solution: find y_p, solve the homogeneous equation and add them together.**
     Example: $y' + y = x$, $xy' + 2y = \sin x$, etc.

   - We may have another type we could solve, though in general nonlinear: separable equations.

     $$y' = G(x)H(y) \Rightarrow \int \frac{dy}{H(y)} = \int G(x)dx$$

     Example: $y' = 2xy^2$, $y' = -y + 2$ etc.

2 Theories for linear equations(Try to understand)

1. Concepts:

   - We say an operator T is linear if $T(y_1 + y_2) = T(y_1) + T(y_2)$ and $T(cy) = cT(y)$ which means you can pull out sum and constants. Equation $T(y) = q(x)$ is called linear equation if T is linear operator.

     Example: $x^2y'' - xy' + 3y = x$, $y' - 2y = x$, $y'' - e^x y + y = 0$ are linear equations while $y' - y^2 = 0, x^2y'' - y + \sin(y) = x$ are nonlinear equations.

   - We say a LINEAR equation is homogeneous if there is no term that doesn't contain y. Otherwise, we say the linear equation is inhomogeneous. $y' + xy = 2$ is inhomogeneous while $y'' - y = 0$ is homogeneous.

2. General theory:

   - (***existence and uniqueness theorem). For a n-th order equation, if the coefficients are good, there is one and only one solution which satisfies n given initial conditions: $y(0) = y_0, y'(0) = v_0, \ldots, y^{(n-1)}(0) = \ldots$.
   - For a homogeneous linear equation, $T(y) = 0$, if it's n-th order, we can find the whole list of solutions by finding n independent basis solutions and superposing them. That means if $y_1, y_2, \ldots, y_n$ are independent solutions, we can form the whole list of solutions by writing

     $$y = C_1y_1 + C_2y_2 + \ldots + C_ny_n$$

     This is easy to understand because T is linear. $T(C_1y_1 + C_2y_2) = C_1T(y_1) + C_2T(y_2)$, since both $T(y_1)$ and $T(y_2)$ are zero, the left hand side is zero.

     To understand why it is a complete list, one should use the existence and uniqueness theorem to argue, which is your homework.

   - For an inhomogeneous linear equation $T(y) = q$. One could compute a particular solution y_p; solve the corresponding homogeneous equation to get y_h and then add them together, $y = y_p + y_h$. (y_h yields zero while y_p takes charge of q. We need y_h because we'll keep the whole list complete.)

3 Second order equations and applications

1. The simplest second order equation: constant acceleration

$$x''(t) = a \quad x(t_0) = x_0, v(t_0) = x'(t_0) = v_0$$

   One could solve it by integrating directly: $v(t) = a(t - t_0) + v_0$ and $x(t) = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + x_0$.

   One could solve it by using the characteristic polynomial $r^2 = 0$ too. The two solutions are $e^{0t} = 1$ and $te^{0t} = t$. The particular solution is $\frac{1}{2}at^2$. Then, $x = C_1 * 1 + C_2 * t + \frac{1}{2}at^2$. One corollary that might be useful is $v_{final}^2 - v_{initial}^2 = 2as$ which can be obtained by multiplying x' in the equation and integrating.

2. Reducible second order equation: $y'' + p(x)y' = q(x)$. You just let $v = y'$ and then you have a first order equation about v. You can then solve v easily. y is obtained by integrating further.

   $p(x)$ may not be a constant. If it's not constant, you must use the method here. If p is a constant, you can use characteristic polynomial too besides what's mentioned here.

   The *application* of this kind of equation includes falling in air with resistance or a moving car with both engine and air resistance.

   The equation is in general $x'' = a - \rho x'(v' = a - \rho v)$ or $x'' = -g - \rho x'$.

3. General second order linear equation with **constant** coefficients.

$$y'' + Ay' + By = q(x)$$

   where A, B are constants.

   (a) Solve the characteristic equation $r^2 + Ar + B = 0$.
   If it has two real roots r_1, r_2, then the solution to homogeneous equation is

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}$$

   If it has a repeated root r, the solution to homogeneous is

$$y(x) = C_1e^{rx} + C_2xe^{rx}$$

   For repeated root, you can always multiply x to get other basis solutions.

If you have conjugate complex roots $\alpha \pm \beta i$. The two complex solutions $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ yield two real solutions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$. You thus have

$$y(x) = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

(b) If $q = 0$, you are done after step (a). Otherwise, you should find a particular solution y_p that solves the equation. There are some general rules for guessing, which we didn't cover.

(c) Adding y_p and the solutions to homogeneous part yields the full list of general solutions.

You can determine C_1, C_2 by using initial conditions easily.

4. Applications

- Undamped mass/spring system.

$$mx'' + kx = 0 \quad x'' + \omega_0^2 x = 0$$

The characteristic polynomial $r^2 + \omega_0^2 = 0$ yields $r = \pm i\omega_0$, implying that

$$\begin{aligned} x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) &= \sqrt{C_1^2 + C_2^2}\sin(\omega_0 t + \phi) \\ &= \sqrt{C_1^2 + C_2^2}\cos(\omega_0 t - \alpha) \end{aligned}$$

Here $A = \sqrt{C_1^2 + C_2^2}$ is called the amplitude and $\omega_0 = \sqrt{k/m}$ is the frequency.

- Damped mass/spring, or spring-mass-dashpot system.

$$mx'' + cx' + kx = 0 \quad x'' + 2px' + \omega_0^2 x = 0$$

where we introduce notation $p = c/(2m)$ for later convenience.

(a) If $c^2 - 4km < 0$ or $p < \omega_0$, then characteristic equation has two complex conjugate roots $r = -p \pm i\omega_1$ where $\omega_1 = \sqrt{\omega_0^2 - p^2}$ is the pseudo-frequency. Since c is less than some value, we still have oscillation due to sin and cos. That means the resistance is not significant and this case is called underdamped.

$$x(t) = e^{-pt}(C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)) = Ce^{-pt}\cos(\omega_1 t - \alpha)$$

$C = \sqrt{C_1^2 + C_2^2}$. Ce^{-pt} and $-Ce^{-pt}$ are the two envelopes.

(b) If $c^2 - 4km = 0$ or $p = \omega_0$, we have repeated root $-p$. The system is called critically damped.

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}$$

(c) If $c^2 - 4km > 0$ or $p > \omega_0$, we have two real roots r_1, r_2. The system has no oscillation. It's called overdamped because c is so large that it gets rid of the oscillation.

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

4 Numerical Method: Euler's method

For general equation $y' = f(x, y)$ where we can or can't find formulas for y, we may do numerics to see how the solution looks like. Given $y(0) = y_0$, we can use the formula

$$y_{n+1} = y_n + h * f(x_n, y_n)$$

to get an approximation of the values of y at the target points.