Summary

In this summary, I usually use $x$ as the variable (instead of $t$). In the spring-mass system, I may use $x$ for the function

1 Previous knowledge

1. Set notations ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \in, \subseteq, \notin$ etc), set builder notation; Function notations (arrow notation); Integral formulas, integration by parts, trig substitution etc; fundamental theorem, Mean value theorem. (For this part, just need to know them)

2. Techniques for first order equations

Terminology: Order means the order of highest derivative

- First order linear equation

$$y'(x) + p(x)y = q(x)$$

Notice that this equation may have non-constant coefficients (e.g. $y' + xy = \sin x$ is linear but coefficients are non-constant.) The technique in previous course is integrating factor:

$$\rho(x) = e^{\int p(x)dx}$$

Multiplying the factor, one obtains

$$\rho(x)(y' + p(x)y) = (\rho(x)y)' = \rho(x)q(x)$$

Integrating yields the GENERAL solution $y = \frac{1}{\rho(x)} \int \rho(x)q(x)dx$.

One can also use the theory below to find the general solution: find $y_p$, solve the homogeneous equation and add them together.

Example: $y' + y = x, xy' + 2y = \sin x, \text{ etc.}$

- We may have another type we could solve, though in general nonlinear: separable equations.

$$y' = G(x)H(y) \Rightarrow \int \frac{dy}{H(y)} = \int G(x)dx$$

Example: $y' = 2xy^2, y' = -y + 2 \text{ etc.}$
2 Theories for linear equations (Try to understand)

1. Concepts:

- We say an operator $T$ is linear if $T(y_1 + y_2) = T(y_1) + T(y_2)$ and $T(cy) = cT(y)$ which means you can pull out sum and constants. Equation $T(y) = q(x)$ is called linear equation if $T$ is linear operator.

Example: $x^2y'' - xy' + 3y = x$, $y' - 2y = x$, $y'' - e^x y + y = 0$ are linear equations while $y' - y^2 = 0$, $x^2y'' - y + \sin(y) = x$ are nonlinear equations.

- We say a LINEAR equation is homogeneous if there is no term that doesn’t contain $y$. Otherwise, we say the linear equation is inhomogeneous. $y' + xy = 2$ is inhomogeneous while $y'' - y = 0$ is homogeneous.

2. General theory:

- (**existence and uniqueness theorem). For a $n$-th order equation, if the coefficients are good, there is one and only one solution which satisfies $n$ given initial conditions: $y(0) = y_0$, $y'(0) = v_0$, ..., $y^{(n-1)}(0) = \ldots$.

- For a homogeneous linear equation, $T(y) = 0$, if it’s $n$-th order, we can find the whole list of solutions by finding $n$ independent basis solutions and superposing them. That means if $y_1, y_2, \ldots, y_n$ are independent solutions, we can form the whole list of solutions by writing

$$y = C_1y_1 + C_2y_2 + \ldots + C_ny_n$$

This is easy to understand because $T$ is linear. $T(C_1y_1 + C_2y_2) = C_1T(y_1) + C_2T(y_2)$, since both $T(y_1)$ and $T(y_2)$ are zero, the left hand side is zero.

To understand why it is a complete list, one should use the existence and uniqueness theorem to argue, which is your homework.

- For an inhomogeneous linear equation $T(y) = q$. One could compute a particular solution $y_p$; solve the corresponding homogeneous equation to get $y_h$ and then add them together, $y = y_p + y_h$. ($y_h$ yields zero while $y_p$ takes charge of $q$. We need $y_h$ because we'll keep the whole list complete.)
3 Second order equations and applications

1. The simplest second order equation: constant acceleration

\[ x''(t) = a \quad x(t_0) = x_0, v(t_0) = v_0 \]

One could solve it by integrating directly: \( v(t) = a(t - t_0) + v_0 \) and
\[ x(t) = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + x_0. \]

One could solve it by using the characteristic polynomial \( r^2 = 0 \) too.
The two solutions are \( e^{0t} = 1 \) and \( te^{0t} = t \). The particular solution is \( \frac{1}{2}at^2 \). Then, \( x = C_1 + C_2 t + \frac{1}{2}at^2 \). One corollary that might be useful is \( v_{\text{final}}^2 - v_{\text{initial}}^2 = 2as \) which can be obtained by multiplying \( x' \) in the equation and integrating.

2. Reducible second order equation: \( y'' + p(x)y' = q(x) \). You just let \( v = y' \) and then you have a first order equation about \( v \). You can then solve \( v \) easily. \( y \) is obtained by integrating further.

\( p(x) \) may not be a constant. If it’s not constant, you must use the method here. If \( p \) is a constant, you can use characteristic polynomial too besides what’s mentioned here.

The application of this kind of equation includes falling in air with resistance or a moving car with both engine and air resistance.

The equation is in general \( x'' = a - \rho x'(v' = a - \rho v) \) or \( x'' = -g - \rho x' \).

3. General second order linear equation with constant coefficients.

\[ y'' + Ay' + By = q(x) \]

where \( A, B \) are constants.

(a) Solve the characteristic equation \( r^2 + Ar + B = 0 \).

If it has two real roots \( r_1, r_2 \), then the solution to homogeneous equation is
\[ y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \]

If it has a repeated root \( r \), the solution to homogeneous is
\[ y(x) = C_1 e^{rx} + C_2 xe^{rx} \]

For repeated root, you can always multiply \( x \) to get other basis solutions.
If you have conjugate complex roots $\alpha \pm \beta i$. The two complex solutions $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ yield two real solutions $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$. You thus have

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

(b) If $q = 0$, you are done after step (a). Otherwise, you should find a particular solution $y_p$ that solves the equation. There are some general rules for guessing, which we didn’t cover.

(c) Adding $y_p$ and the solutions to homogeneous part yields the full list of general solutions.

You can determine $C_1, C_2$ by using initial conditions easily.

4. Applications

- Undamped mass/spring system.

$$mx'' + kx = 0 \quad x'' + \omega_0^2 x = 0$$

The characteristic polynomial $r^2 + \omega_0^2 = 0$ yields $r = \pm i \omega_0$, implying that

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = \sqrt{C_1^2 + C_2^2} \sin(\omega_0 t + \phi)$$

$$= \sqrt{C_1^2 + C_2^2} \cos(\omega_0 t - \alpha)$$

Here $A = \sqrt{C_1^2 + C_2^2}$ is called the amplitude and $\omega_0 = \sqrt{k/m}$ is the frequency.

- Damped mass/spring, or spring-mass-dashpot system.

$$mx'' + cx' + kx = 0 \quad x'' + 2px' + \omega_0^2 x = 0$$

where we introduce notation $p = c/(2m)$ for later convenience.

(a) If $c^2 - 4km < 0$ or $p < \omega_0$, then characteristic equation has two complex conjugate roots $r = -p \pm i \omega_1$ where $\omega_1 = \sqrt{\omega_0^2 - p^2}$ is the pseudo-frequency. Since $c$ is less than some value, we still have oscillation due to sin and cos. That means the resistance is not significant and this case is called under-damped.

$$x(t) = e^{-pt}(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)) = Ce^{-pt} \cos(\omega_1 t - \alpha)$$

$$C = \sqrt{C_1^2 + C_2^2}. \, Ce^{-pt} \text{ and } -Ce^{-pt} \text{ are the two envelopes.}$$
(b) If \( c^2 - 4km = 0 \) or \( p = \omega_0 \), we have repeated root \(-p\). The system is called critically damped.

\[
x(t) = C_1 e^{-pt} + C_2 te^{-pt}
\]

(c) If \( c^2 - 4km > 0 \) or \( p > \omega_0 \), we have two real roots \( r_1, r_2 \). The system has no oscillation. It’s called overdamped because \( c \) is so large that it gets rid of the oscillation.

\[
x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}
\]

4 Numerical Method: Euler’s method

For general equation \( y' = f(x, y) \) where we can or can’t find formulas for \( y \), we may do numerics to see how the solution looks like. Given \( y(0) = y_0 \), we can use the formula

\[
y_{n+1} = y_n + h \cdot f(x_n, y_n)
\]

to get an approximation of the values of \( y \) at the target points.