The system is

$$\begin{cases} x' = x + 5y + 3z + 3 \\ y' = 3y + 2z - 2 \\ z' = -7x + 4y + 5z + 1 \end{cases}$$

- Check if there is any constant solution, if no, try linear, etc.

  If $x, y, z = const$, then the derivatives are all zero. Therefore you get the equations

$$\begin{cases} x + 5y + 3z = -3 \\ 3y + 2z = 2 \\ -7x + 4y + 5z = -1 \end{cases}$$

  Write out the augmented matrix as

$$\begin{bmatrix} 1 & 5 & 3 & -3 \\ 0 & 3 & 2 & 2 \\ -7 & 4 & 5 & -1 \end{bmatrix}$$

  Doing Gauss elimination or using $rref$ in your calculator, your last row is something like [0 0 0 4](you may get other nonzero number instead of 4). Therefore, this linear system of algebraic equations has no solutions and the 1st order linear differential system doesn't have **constant** solutions.(it must have nonconstant solutions then by existence/uniqueness).

  Now, you try linear functions. $x = a_1 t + b_1, y = a_2 t + b_2, z = a_3 t + b_3$. Plugging in, you have

$$\begin{cases} a_1 = (a_1 t + b_1) + 5(a_2 t + b_2) + 3(a_3 t + b_3) + 3 \\ a_2 = 3(a_2 t + b_2) + 2(a_3 t + b_3) - 2 \\ a_3 = -7(a_1 t + b_1) + 4(a_2 t + b_2) + 5(a_3 t + b_3) + 1 \end{cases}$$

  Then, comparing the coefficients, you get the following 6 equations

$$\begin{cases} 0 = a_1 + 5a_2 + 3a_3 \\ a_1 = b_1 + 5b_2 + 3b_3 + 3 \\ 0 = 3a_2 + 2a_3 \\ a_2 = 3b_2 + 2b_3 - 2 \\ 0 = -7a_1 + 4a_2 + 5a_3 \\ a_3 = -7b_1 + 4b_2 + 5b_3 + 1 \end{cases}$$

Then, you get the following augmented matrix

$$\begin{array}{ccccccc}
a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & \\
1 & 5 & 3 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 5 & 3 & -3 \\
0 & 3 & 2 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 3 & 2 & 2 \\
-7 & 4 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -7 & 4 & 5 & -1
\end{array}$$

Solving this using $rref$, you'll see that this is consistent. The last variable can be picked as free variable. We thus pick $b_3 = 0$. Then, $a_1 = 1.333, a_2 = -2.667, a_3 = 4, b_1 = -0.5555, b_2 = -0.2222$(probably $a_1 = 4/3, a_2 = -8/3, a_3 = 4, b_1 = -5/9, b_2 = -2/9$). This means we have found a solution $(x(t), y(t), z(t)) = (1.333t - 0.555,\ -2.667t - 0.222,\ 4t)$

- Find a solution by applying a uniform method.

  We know that the degree of a particular solution which is a bunch of polynomials won't exceed $n = 3$. We therefore can try 3rd order polynomials. Namely, you plug in $x = a_1t^3 + b_1t^2 + c_1t + d_1, y = a_2t^3 + b_2t^2 + c_2t + d_2, z = a_3t^3 + b_3t^2 + c_3t + d_3$.

  Then, comparing coefficients, you get a $12 * 12$ system. The method is the same as the one in lecture. Read the solution online and finish this. I really don't want to do this here.

- Verify that $(x_1, y_1, z_1) = (-0.2673, 0.5345, -0.8018)$ is a solution to the homogeneous system.

  The homogeneous system is

$$\begin{cases} x' = x + 5y + 3z \\ y' = 3y + 2z \\ z' = -7x + 4y + 5z \end{cases}$$

  Plug the expressions in, and you see the first equation requires $0 = -0.2673 + 5 * 0.5345 + 3 * (-0.8018) \approx -2 \times 10^{-4}$. Within the numerical error bound, this is true. Similarly you can verify that the other two equations also hold. Therefore, this is a solution to the homogeneous system.

  Remark: The homogeneous system has an eigenvalue $\lambda = 0$ and this solution here is one corresponding eigenvector. The solution is therefore $e^{0t} * v = v$

- Let $\alpha = 4.5, \beta = 3.9686$. Verify that

$$\begin{cases} x_2 = 0.1203e^{\alpha t}\cos\beta t + 0.5909e^{\alpha t}\sin\beta t \\ y_2 = 0.1203e^{\alpha t}\cos\beta t + 0.3182e^{\alpha t}\sin\beta t \\ z_2 = 0.7216e^{\alpha t}\cos\beta t \end{cases}$$

  is a solution to the homogeneous system.

  Of course, to verify, you simply plug in to the homogeneous system!. If you don't know how to verify, well, I think you'll fail this course!

  Another solution is

$$\begin{cases} x_3 = -0.5909e^{\alpha t}\cos\beta t + 0.1203e^{\alpha t}\sin\beta t \\ y_3 = -0.3182e^{\alpha t}\cos\beta t + 0.1203e^{\alpha t}\sin\beta t \\ z_3 = 0.7216e^{\alpha t}\sin\beta t \end{cases}$$

  Also to verify, just plug in.

  Remark: these two solutions come from the eigenvalues $\lambda = \alpha \pm i\beta$. The corresponding eigenvectors are $v_1 \pm iv_2$. Then two solutions are real and imaginary parts of $e^{(\alpha+i\beta)t}(v_1 + iv_2)$

- Check independence. The Wronskian system at $t = 0$ is

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & 0 \\ y_1(0) & y_2(0) & y_3(0) & 0 \\ z_1(0) & z_2(0) & z_3(0) & 0 \end{bmatrix}$$

  You fill in numbers and get

$$\begin{bmatrix} -0.2673 & 0.1203 & -0.5909 & 0 \\ 0.5345 & 0.1203 & -0.3182 & 0 \\ -0.8018 & 0.7216 & 0 & 0 \end{bmatrix}$$

  This system only has the zero solution. Therefore, the three solutions of the homogeneous system are linearly independent.

- Well, some system that has nontrivial solutions can be perturbed by numerical errors to a system that only has zero solution. How can we check if the system originally is singular(has nontrivial soln) or not? One fact is that if you perturb a singular system with small numerical errors, the system is still near singular.

  Within your knowledge, one way is to look at the determinant. If the determinant is very small, the original is probably singular. However, there is some issue for just looking at the determinant. A more rigorous way is to look at the so-called condition number. You can see this in numerical analysis course.

- First of all, you should know the dimension of the solution space of the homogeneous system is $n = 3$ by dimension theorem. Secondly, above you have seen that the three solutions are independent. Therefore, the linear combinations of the three solutions cover the solution space of the homogeneous system. Adding the particular solution yields what you need. The complete list of solutions is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \end{pmatrix} + C_2 \begin{pmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{pmatrix} + C_3 \begin{pmatrix} x_3(t) \\ y_3(t) \\ z_3(t) \end{pmatrix} + \begin{pmatrix} 1.333t - 0.555 \\ -2.667t - 0.222 \\ 4t \end{pmatrix}$$

  with C_1, C_2, C_3 arbitrary scalars.

- We have three undetermined coefficients. Then, if you have initial conditions for x, y, z, you can determine C_1, C_2, C_3 uniquely. This actually verifies the existence/uniqueness in this specific example.