Material for discussion (Summary of differential equations covered in Math222)

1 First order equations

By 'first order', we mean the highest order of derivatives in the equation is first order.

1.1 Linear equations

One can check the operation defined as follows is linear:

$$T(y) = \frac{d}{dx}y + p(x)y$$

Recall 'linear' means $T(y_1 + y_2) = T(y_1) + T(y_2)$ and $T(cy) = cT(y)$.

Thus, first order linear equations should have the following form:

$$y'(x) + p(x)y = q(x)$$

When $q(x) = 0$, it's homogeneous and the 'superposition principle' applies(namely for any two solutions y_1, y_2, the linear combination $C_1y_1 + C_2y_2$ or ay_1+by_2 is also a solution). When $q(x) \neq 0$, it's inhomogeneous, and one can find any particular solution y_p, solve the corresponding homogeneous solution to get $y_h = Ce^{-\int p dx}$ and have the general solution

$$y = y_p + Ce^{-\int p dx}$$

However, for first order linear equation, we have another method called **integrating factor** method. The integrating factor is

$$\mu = e^{\int p dx}$$

Multiply μ on the equation and one has

$$(\mu y)' = q(x)\mu(x)$$

Then, one can simply find y by integrating.

Example 1

$$y' = 2y + x^2$$

Find the general solution.

Method 1:

The equation is linear $y' - 2y = x^2$.

One may simply guess that a particular solution is a quadratic function. Then, plug in $y = ax^2 + bx + c$ and we have

$$y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}$$

The solution to the homogeneous part is

$$y' - 2y = 0 \Rightarrow y = Ce^{2x}$$

Therefore, the full list of solutions is

$$y = Ce^{2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}$$

Method 2. The integrating factor is

$$\mu = e^{\int(-2)dx} = e^{-2x}$$

Then, the equation can be reduced to

$$(e^{-2x}y)' = x^2e^{-2x}$$

Integrating by parts, we have

$$e^{-2x}y = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

The same solution.

1.2 Separable equations

We may also come across nonlinear differential equations which don't allow 'superposition principle'. One specific type is the separable equations and we are able to solve them. (Note that some linear equations could also be separable.)

The following type is separable

$$y'(x) = H(y)G(x)$$

which could be solved by doing the integrals

$$\int \frac{dy}{H(y)} = \int G(x)dx$$

Example 2 $y' = 2y + 3$

This is both linear and separable. If we use the technique for separable equations, we have

$$\frac{dy}{2y+3} = dx$$

Integrating, we can solve y.

Example 3 $y'(x) - xy^2 = x$

Notice that this is not linear. This can be rearranged to $y'(x) = xy^2 + x = (1 + y^2)x$. We then have

$$\frac{dy}{1+y^2} = xdx$$

which gives $\arctan y = \frac{1}{2}x^2 + C$ and thus $y = \tan(\frac{1}{2}x^2 + C)$.

1.3 Other special types of nonlinear equations

Bernoulli equations, Riccati equations, exact equations etc. These equations are not covered in Math222. You can read our textbook for Bernoulli and exact equations.

2 Second order linear equation of constant coefficients

In Math222, we only covered linear equations of second order with constant coefficients, namely

$$ay'' + by' + cy = q(x)$$

When $q(x) = 0$, it's homogeneous and when $q(x) \neq 0$, it's inhomogeneous.

The homogeneous part is

$$ay'' + by' + cy = 0$$

The equation being linear and homogeneous, the 'superposition principle' applies. To find the solution, we use the characteristic polynomial and superposition principle.

Example 4 $y'' + 4y = 0$

This is the equation in spring-mass system. The characteristic polynomial is

$$\lambda^2 + 4 = 0$$

We solve that $\lambda = \pm 2i$. Therefore, we have $e^{2ix} = \cos(2x) + i\sin(2x)$ and $e^{-2ix} = \cos(2x) - i\sin(2x)$ as the solutions. Then, by superposition principle, we know $\cos(2x) = \frac{1}{2}(e^{2ix} + e^{-2ix})$ and $\sin(2x) = \frac{1}{2i}(e^{2ix} - e^{-2ix})$ are two solutions. Then, we find by superposition the general solutions

$$y(x) = C_1\cos(2x) + C_2\sin(2x)$$

Example 5 *Find the general solutions to* $y'' - 2y' - 3y = x$

This is inhomogeneous. We can find the solutions to homogeneous part first:

$$y'' - 2y' - 3y = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = 3, -1$$

By superposition, the general solution to homogeneous part is

$$y = C_1e^{3x} + C_2e^{-x}$$

For the particular part, we guess $y_p = mx + n$ and find one particular solution $y_p = -\frac{1}{3}x + \frac{2}{9}$. The list of solutions is

$$y = C_1e^{3x} + C_2e^{-x} - \frac{1}{3}x + \frac{2}{9}$$

Non-constant linear equations, some special types

These equations are hard. For example, the simplest equation in this category one can think of $y'' + xy = 0$ is already very hard. However, there is a type called 'Euler differential equations' that can be solved. Those who are interested can find this out online.