Consider the system of equations, where $x_1 = x_1(t), x_2 = x_2(t)$ are two unknown functions:

$$\begin{cases} x_1' - 3x_1 + 4x_2 = 0 \\ x_2' - x_1 + x_2 = 0 \end{cases}$$

Assume that the two functions satisfy the initial conditions $x_1(0) = 1$, $x_2(0) = 2$. Please find the function $x_1(t)$.

We define the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

and the system of equations can be written as $x' = Ax$ with

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

Some people didn't change the signs for the coefficients when moving them to right. I didn't penalize too much in this quiz.

The characteristic equation

$$det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

Hence there is a repeated eigenvalue $\lambda = 1$. We solve $(A - 1 * I)\xi = 0$:

$$A - I = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

One solution is $x^{(1)} = e^t\xi$.This is not the x_1 in the problem. We need another solution. We solve the generalized eigenvector problem $(A - I)\eta = \xi$.

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \eta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu\xi.$$

We can simply choose $\mu = 0$ to get the generalized eigenvector and have another solution

$$x^{(2)} = te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The general solution is $x = C_1x^{(1)} + C_2x^{(2)} = \Psi(t)c$. At $t = 0$, we have

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow c = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Hence, $x_1(t) = 2 * (2e^t) + (-3)(2te^t + e^t) = e^t - 6te^t$.

(Bonus 1. 2 pts). Consider $x' = Ax$ where A is the identity matrix I_2. Is the eigenvalue repeated? Can you find a fundamental matrix for this system?

It's easy to see that the characteristic equation is $(\lambda - 1)^2 = 0$ and hence $\lambda = 1$ is repeated. We have $A - 1 * I$ to be the zero matrix. Hence, any nonzero vector is an eigenvector. We thus can have two independent eigenvector $\xi^{(1)} = (1, 0)^T$ and $\xi^{(2)} = (0, 1)^T$. We thus have two independent solutions $x^{(1)} = e^t\xi^{(1)}$ and $x^{(2)} = e^t\xi^{(2)}$ which will make a fundamental matrix. Note that the fundamental matrix always exists. We need two solutions. In this case, there is no generalized eigenvector η that is not an eigenvector.

(Bonus 2. 2+4 pts). Suppose matrix A is 2×2. It has two eigenvalues $1, 2$ and the corresponding eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
(a). Write A into the form $A = PDP^{-1}$ where D is a diagonal matrix.(Hint: $P = [\xi^{(1)}, \xi^{(2)}]$). (b). A fundamental matrix for the system $x' = Ax$ is $\Psi(t) = \exp(At)$. Use the form obtained to compute this fundamental matrix.

By the eigenvalue definition, we have $A\xi^{(1)} = \xi^{(1)}$ and $A\xi^{(2)} = 2\xi^{(2)}$. If P is defined like that, we have

$$AP = [\xi^{(1)}, 2\xi^{(2)}] = P\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow A = PDP^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{pmatrix}$$

We thus have a fundamental matrix

$$\Psi(t) = \exp(PDtP^{-1}) = P\exp(Dt)P^{-1} = P\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}P^{-1}.$$

(Bonus 3. 4 pts). Consider the linear system $x' = Ax$ where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the general solution.

The eigenvalue is easy to see $\lambda = 1$ which is repeated. We can have two independent eigenvectors $\xi^{(1)} = (1, 0, 0)^T$ and $\xi^{(2)} = (0, 0, 1)^T$. Hence, we can construct two solutions from them. However, we need three solutions. Another one must be from the generalized eigenvector η. In this case, only $\xi^{(1)}$ gives a generalized eigenvector

$$(A - I)\eta = \xi^{(1)} \Rightarrow \eta = (0, 1, 0).$$

Hence, $x = C_1e^t\xi^{(1)} + C_2(te^t\xi^{(1)} + e^t\eta) + C_3e^t\xi^{(2)}$.

Consider the system of equations, where $x_1 = x_1(t), x_2 = x_2(t)$ are two unknown functions:

$$\begin{cases} x_1' - x_1 + x_2 = 0 \\ x_2' - x_1 - 3x_2 = 0 \end{cases}$$

Assume that the two functions satisfy the initial conditions $x_1(0) = 2, x_2(0) = 1$. Please find the function $x_2(t)$.

We define the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

and the system of equations can be written as $x' = Ax$ with

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Some people didn't change the signs for the coefficients when moving them to right. I didn't penalize too much in this quiz.

The characteristic equation

$$det(A - \lambda I) = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Hence there is a repeated eigenvalue $\lambda = 2$. We solve $(A - 2I)\xi = 0$:

$$A - 2I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

One solution is $x^{(1)} = e^{2t}\xi$. This is not the x_1 in the problem. We need another solution. We solve the generalized eigenvector problem $(A - 2I)\eta = \xi$.

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \mu\xi.$$

We can simply choose $\mu = 0$ to get the generalized eigenvector and have another solution

$$x^{(2)} = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The general solution is $x = C_1 x^{(1)} + C_2 x^{(2)} = \Psi(t)c$. At $t = 0$, we have

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow c = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Hence, $x_2(t) = 2 * (-e^{2t}) + (-3)(-te^{2t} - e^{2t}) = e^{2t} + 3te^{2t}$.

(Bonus 1. 2 pts). Consider $x' = Ax$ where A is the identity matrix I_2. Is the eigenvalue repeated? Can you find a fundamental matrix for this system?

It's easy to see that the characteristic equation is $(\lambda - 1)^2 = 0$ and hence $\lambda = 1$ is repeated. We have $A - 1 * I$ to be the zero matrix. Hence, any nonzero vector is an eigenvector. We thus can have two independent eigenvector $\xi^{(1)} = (1, 0)^T$ and $\xi^{(2)} = (0, 1)^T$. We thus have two independent solutions $x^{(1)} = e^t\xi^{(1)}$ and $x^{(2)} = e^t\xi^{(2)}$ which will make a fundamental matrix. Note that the fundamental matrix always exists. We need two solutions. In this case, there is no generalized eigenvector η that is not an eigenvector.

(Bonus 2. 2+4 pts). Suppose matrix A is 2×2. It has two eigenvalues $1, 2$ and the corresponding eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
(a). Write A into the form $A = PDP^{-1}$ where D is a diagonal matrix.(Hint: $P = [\xi^{(1)}, \xi^{(2)}]$). (b). A fundamental matrix for the system $x' = Ax$ is $\Psi(t) = \exp(At)$. Use the form obtained to compute this fundamental matrix.

By the eigenvalue definition, we have $A\xi^{(1)} = \xi^{(1)}$ and $A\xi^{(2)} = 2\xi^{(2)}$. If P is defined like that, we have

$$AP = [\xi^{(1)}, 2\xi^{(2)}] = P\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow A = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}$$

We thus have a fundamental matrix

$$\Psi(t) = \exp(PDtP^{-1}) = P\exp(Dt)P^{-1} = P\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}P^{-1}.$$

(Bonus 3. 4 pts). Consider the linear system $x' = Ax$ where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the general solution.

The eigenvalue is easy to see $\lambda = 1$ which is repeated. We can have two independent eigenvectors $\xi^{(1)} = (1, 0, 0)^T$ and $\xi^{(2)} = (0, 0, 1)^T$. Hence, we can construct two solutions from them. However, we need three solutions. Another one must be from the generalized eigenvector η. In this case, only $\xi^{(1)}$ gives a generalized eigenvector

$$(A - I)\eta = \xi^{(1)} \Rightarrow \eta = (0, 1, 0).$$

Hence, $x = C_1e^t\xi^{(1)} + C_2(te^t\xi^{(1)} + e^t\eta) + C_3e^t\xi^{(2)}$.