Consider the system of equations, where \( x_1 = x_1(t) \), \( x_2 = x_2(t) \) are two unknown functions:

\[
\begin{align*}
    x_1' - 3x_1 + 4x_2 &= 0 \\
    x_2' - x_1 + x_2 &= 0
\end{align*}
\]

Assume that the two functions satisfy the initial conditions \( x_1(0) = 1, \) \( x_2(0) = 2 \). Please find the function \( x_1(t) \).

We define the vector

\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \]

and the system of equations can be written as \( x' = Ax \) with

\[ A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}. \]

Some people didn't change the signs for the coefficients when moving them to right. I didn't penalize too much in this quiz.

The characteristic equation

\[ det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0. \]

Hence there is a repeated eigenvalue \( \lambda = 1 \). We solve \((A - 1* I)\xi = 0:\)

\[ A - I = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

One solution is \( x^{(1)} = e^t\xi \). This is not the \( x_1 \) in the problem. We need another solution. We solve the generalized eigenvector problem \((A - I)\eta = \xi. \)

\[ \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \eta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu\xi. \]

We can simply choose \( \mu = 0 \) to get the generalized eigenvector and have another solution

\[ x^{(2)} = te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

The general solution is \( x = C_1x^{(1)} + C_2x^{(2)} = \Psi(t)c. \) At \( t = 0 \), we have

\[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow c = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}. \]

Hence, \( x_1(t) = 2 * (2e^t) + (-3)(2te^t + e^t) = e^t - 6te^t. \)
(Bonus 1. 2 pts). Consider $x' = Ax$ where $A$ is the identity matrix $I_2$. Is the eigenvalue repeated? Can you find a fundamental matrix for this system?

It’s easy to see that the characteristic equation is $(\lambda - 1)^2 = 0$ and hence $\lambda = 1$ is repeated. We have $A - 1 \times I$ to be the zero matrix. Hence, any nonzero vector is an eigenvector. We thus can have two independent eigenvector $\xi^{(1)} = (1, 0)^T$ and $\xi^{(2)} = (0, 1)^T$. We thus have two independent solutions $x^{(1)} = e^t \xi^{(1)}$ and $x^{(2)} = e^t \xi^{(2)}$ which will make a fundamental matrix. Note that the fundamental matrix always exists. We need two solutions. In this case, there is no generalized eigenvector $\eta$ that is not an eigenvector.

(Bonus 2. 2+4 pts). Suppose matrix $A$ is $2 \times 2$. It has two eigenvalues 1, 2 and the corresponding eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(a). Write $A$ into the form $A = PD P^{-1}$ where $D$ is a diagonal matrix. (Hint: $P = [\xi^{(1)}, \xi^{(2)}]$). (b). A fundamental matrix for the system $x' = Ax$ is $\Psi(t) = \exp(At)$. Use the form obtained to compute this fundamental matrix.

By the eigenvalue definition, we have $A\xi^{(1)} = \xi^{(1)}$ and $A\xi^{(2)} = 2\xi^{(2)}$. If $P$ is defined like that, we have

$$AP = [\xi^{(1)}, 2\xi^{(2)}] = P \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow A = PD P^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/2 & 1/2 \end{pmatrix}$$

We thus have a fundamental matrix

$$\Psi(t) = \exp(PTDP^{-1}) = P \exp(Dt)P^{-1} = P \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1}.$$ 

(Bonus 3. 4 pts). Consider the linear system $x' = Ax$ where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the general solution.

The eigenvalue is easy to see $\lambda = 1$ which is repeated. We can have two independent eigenvectors $\xi^{(1)} = (1, 0, 0)^T$ and $\xi^{(2)} = (0, 0, 1)^T$. Hence, we can construct two solutions from them. However, we need three solutions. Another one must be from the generalized eigenvector $\eta$. In this case, only $\xi^{(1)}$ gives a generalized eigenvector

$$(A - I)\eta = \xi^{(1)} \Rightarrow \eta = (0, 1, 0).$$

Hence, $x = C_1 e^t \xi^{(1)} + C_2 (te^t \xi^{(1)} + e^t \eta) + C_3 e^t \xi^{(2)}$. 

2
Consider the system of equations, where \( x_1 = x_1(t), x_2 = x_2(t) \) are two unknown functions:

\[
\begin{align*}
    x'_1 - x_1 + x_2 &= 0 \\
    x'_2 - x_1 - 3x_2 &= 0
\end{align*}
\]

Assume that the two functions satisfy the initial conditions \( x_1(0) = 2, x_2(0) = 1 \). Please find the function \( x_2(t) \).

We define the vector

\[
    x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

and the system of equations can be written as \( x' = Ax \) with

\[
    A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.
\]

Some people didn’t change the signs for the coefficients when moving them to right. I didn’t penalize too much in this quiz.

The characteristic equation

\[
    \det(A - \lambda I) = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.
\]

Hence there is a repeated eigenvalue \( \lambda = 2 \). We solve \( (A - 2I)x = 0 \):

\[
    A - 2I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

One solution is \( x^{(1)} = e^{2t}\xi \). This is not the \( x_1 \) in the problem. We need another solution. We solve the generalized eigenvector problem \( (A - 2I)\eta = \xi \).

\[
    \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \mu\xi.
\]

We can simply choose \( \mu = 0 \) to get the generalized eigenvector and have another solution

\[
    x^{(2)} = te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

The general solution is \( x = C_1 x^{(1)} + C_2 x^{(2)} = \Psi(t)c \). At \( t = 0 \), we have

\[
    \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow c = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.
\]

Hence, \( x_2(t) = 2*(-e^{2t}) + (-3)(-te^{2t} - e^{2t}) = e^{2t} + 3te^{2t} \).
(Bonus 1. 2 pts). Consider $x' = Ax$ where $A$ is the identity matrix $I_2$. Is the eigenvalue repeated? Can you find a fundamental matrix for this system?

It’s easy to see that the characteristic equation is $(\lambda - 1)^2 = 0$ and hence $\lambda = 1$ is repeated. We have $A - 1 \times I$ to be the zero matrix. Hence, any nonzero vector is an eigenvector. We thus can have two independent eigenvector $\xi^{(1)} = (1, 0)^T$ and $\xi^{(2)} = (0, 1)^T$. We thus have two independent solutions $x^{(1)} = e^t \xi^{(1)}$ and $x^{(2)} = e^t \xi^{(2)}$ which will make a fundamental matrix. Note that the fundamental matrix always exists. We need two solutions. In this case, there is no generalized eigenvector $\eta$ that is not an eigenvector.

(Bonus 2. 2+4 pts). Suppose matrix $A$ is $2 \times 2$. It has two eigenvalues $1, 2$ and the corresponding eigenvectors are $\xi^{(1)} = \left( \begin{array}{c} 1 \\ 2 \end{array} \right)$ and $\xi^{(2)} = \left( \begin{array}{c} 1 \\ -1 \end{array} \right)$.

(a). Write $A$ into the form $A = PDP^{-1}$ where $D$ is a diagonal matrix. (Hint: $P = [\xi^{(1)}, \xi^{(2)}]$). (b). A fundamental matrix for the system $x' = Ax$ is $\Psi(t) = \exp(At)$. Use the form obtained to compute this fundamental matrix.

By the eigenvalue definition, we have $A\xi^{(1)} = \xi^{(1)}$ and $A\xi^{(2)} = 2\xi^{(2)}$. If $P$ is defined like that, we have

$$AP = [\xi^{(1)}, 2\xi^{(2)}] = P \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \Rightarrow A = PDP^{-1} = \left( \begin{array}{cc} 1 & 1 \\ 2 & -1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \left( \begin{array}{cc} 1/3 & 1/3 \\ 2/3 & -1/3 \end{array} \right)$$

We thus have a fundamental matrix

$$\Psi(t) = \exp(PDtP^{-1}) = P \exp(Dt)P^{-1} = P \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{2t} \end{array} \right) P^{-1}.$$