1 Version 1

Consider the equation where $y = y(t)$:

$$L(y) = y'' - 5y' + 6y = 0$$

Answer the questions below.

- (a). (3 pts) If $y_1(t)$ and $y_2(t)$ are two solutions, verify that $y(t) = \sqrt{2}y_1(t) + \pi y_2(t)$ is also a solution.

  Since y_1 is a solution, then if you plug y_1 into the equation, the left hand side equals the right hand side, which is zero. In other words, $L(y_1) = y_1'' - 5y' + 6y_1 = 0$. Similarly, $L(y_2) = 0$.

  To verify y, we simply plug it into the left hand side of the equation and have

  $$L(y) = L(\sqrt{2}y_1 + \pi y_2) = \sqrt{2}L(y_1) + \pi L(y_2) = \sqrt{2} * 0 + \pi * 0 = 0$$

  This means y is also a solution.

- (b). (2+5 pts) If $y_1 = e^{2t}$ and $y_2 = e^{3t}$, check that the Wronskian at $t = 0$ is nonzero; Given any solution $\phi(t)$ of the ODE, explain why it can be written as $\phi(t) = C_1 y_1(t) + C_2 y_2(t)$. (Hint: Use the Existence and Uniqueness Theorem as we did in discussion. *Comment: This then explains why* $\{y_1, y_2\}$ *is a fundamental set.*)

  The Wronskian is $W = y_1 y_2' - y_1' y_2 = e^{2t} 3e^{3t} - 2e^{2t} e^{3t}$, which is 1 at $t = 0$.

  Consider the initial conditions of ϕ: $\phi(0), \phi'(0)$. Now, since the **Wronskian is nonzero**, there exist a unique group of $\{C_1, C_2\}$ such that

  $$C_1 y_1(0) + C_2 y_2(0) = \phi(0)$$
  $$C_1 y_1'(0) + C_2 y_2'(0) = \phi'(0)$$

  For this particular choice of C_1, C_2, $C_1 y_1(t) + C_2 y_2(t)$ solves the same IVP as ϕ does. The Existence and Uniqueness theorem says that any IVP has one solution exactly. The only possibility is that ϕ and $C_1 y_1 + C_2 y_2$ must be the same. In other words, ϕ must be in the form as indicated.

2 Version 2

Consider the equation where $y = y(t)$:

$$L(y) = y'' + 4y = 0$$

Answer the questions below.

- (a). (3 pts) If $y_1(t)$ and $y_2(t)$ are two solutions, verify that $y(t) = \sqrt{3}y_1(t) + \pi y_2(t)$ is also a solution.

  Since y_1 is a solution, then if you plug y_1 into the equation, the left hand side equals the right hand side, which is zero. In other words, $L(y_1) = y_1'' + 4y_1 = 0$. Similarly, $L(y_2) = 0$.

  To verify y, we simply plug it into the left hand side of the equation and have

  $$L(y) = L(\sqrt{3}y_1 + \pi y_2) = \sqrt{3}L(y_1) + \pi L(y_2) = \sqrt{3} * 0 + \pi * 0 = 0$$

  This means y is also a solution.

- (b). (2+5 pts) If $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$, check that the Wronskian at $t = 0$ is nonzero; Given any solution $\phi(t)$, explain why it can be written as $\phi(t) = C_1y_1(t) + C_2y_2(t)$. (Hint: Use the Existence and Uniqueness Theorem as we did in discussion. *Comment: This then explains why* $\{y_1, y_2\}$ *is a fundamental set.*)

  The Wronskian is $W = y_1y_2' - y_1'y_2 = \cos(2t) * 2\cos(2t) - (-2\sin(2t))\cos(2t) = 2$, which is nonzero everywhere.

  Consider the initial conditions of ϕ: $\phi(0), \phi'(0)$. Now, since the **Wronskian is nonzero**, there exist a unique group of $\{C_1, C_2\}$ such that

  $$C_1y_1(0) + C_2y_2(0) = \phi(0)$$
  $$C_1y_1'(0) + C_2y_2'(0) = \phi'(0)$$

  For this particular choice of C_1, C_2, $C_1y_1(t) + C_2y_2(t)$ solves the same IVP as ϕ does. The Existence and Uniqueness theorem says that any IVP has one solution exactly. The only possibility is that ϕ and $C_1y_1 + C_2y_2$ must be the same. In other words, ϕ must be in the form as indicated.