#1

$$x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Denote the coefficient matrix by A. First find the fundamental matrix.

$$det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = (\lambda-1)(\lambda+7)+16 = \lambda^2+6\lambda+9 = (\lambda+3)^2 = 0$$

Hence, $\lambda = -3$ which is repeated.

$$A - \lambda I = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}$$

We find one eigenvector and one solution

$$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x^{(1)} = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We need one more solution to construct the fundamental matrix, which is fulfilled by the generalized eigenvector.

$$(A-\lambda I)\eta = \xi \Rightarrow \eta = a\xi + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

Here, for any a, η is fine. We can simply choose $a = 0$. Having η, we can construct another solution

$$x^{(2)} = t3^{-3t}\xi + e^{-3t}\eta = t3^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

It's easy to check that $W(0) = det(x^{(1)}(0), x^{(2)}(0)) \neq 0$ and we therefore have a fundamental matrix

$$\Psi(t) = \begin{pmatrix} e^{-3t} & te^{-3t} + e^{-3t}/4 \\ e^{-3t} & te^{-3t} \end{pmatrix}.$$

For the initial condition, we first compute the combination coefficient:

$$c = \Psi^{-1}(0) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{-1/4} \begin{pmatrix} 0 & -1/4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

We find

$$x = \Psi(t)c = \begin{pmatrix} 3e^{-3t} - 4te^{-3t} \\ 4(1-t)e^{-3t} \end{pmatrix}.$$

7.9.#4

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

Step 1. Solve the homogeneous system. Again, we denote the coefficient by A and compute the characteristic equation

$$det(A - \lambda I) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0.$$

Two eigenvalues $\lambda = 2, \lambda = -3$.

For $\lambda = 2$, we compute

$$A - \lambda I = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}$$

One eigenvector and one solution are given by

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x^{(1)} = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, for $\lambda = -3$:

$$\xi^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \Rightarrow x^{(2)} = e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

We check that $W = det(x^{(1)}, x^{(2)}) \neq 0$ at $t = 0$ and hence a fundamental matrix is given by

$$\Psi(t) = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix}.$$

Step 2. Find a particular solution. The formula is $x_p = \Psi(t)u(t)$ with $u(t) = \int \Psi^{-1}(t)g(t)dt$.

$$\Psi^{-1}(t) = \frac{1}{-5e^{-t}} \begin{pmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4e^{-2t} & e^{-2t} \\ e^{3t} & -e^{3t} \end{pmatrix}.$$

We find

$$u(t) = \int \frac{1}{5} \begin{pmatrix} 4e^{-4t} - 2e^{-t} \\ e^{t} + 2e^{4t} \end{pmatrix} dt = \frac{1}{5} \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ e^{t} + \frac{1}{2}e^{4t} \end{pmatrix}.$$

$$x = \Psi(t)(c + u(t)) = \Psi(t)c + \begin{pmatrix} e^t/2 \\ -e^{-2t} \end{pmatrix}$$