(7.6 #25)

We first of all notice that the units 'H' for inductance, 'F' for capacitance, Ω resistance, 'V' for voltage, 'A' for current, and 's' for time are consistent. Hence, we can ignore the units.

(a). Applying Kirchhoff's Voltage Law(KVL) to the $R_1 - L - C$ loop, we have

$$IR_1 + L\frac{dI}{dt} + V = 0 \Rightarrow I' = -\frac{1}{2}I - \frac{1}{8}V$$

Applying Kirchhoff's Current Law(KCL) to the $L - C$ node(junction), we have

$$I - \frac{V}{R_2} - C\frac{dV}{dt} = 0 \Rightarrow V' = 2I - \frac{1}{2}V$$

In matrix form, we have

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -1/2 & -1/8 \\ 2 & -1/2 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$$

(b). We denote

$$x = \begin{pmatrix} I \\ V \end{pmatrix}$$

from here on. We denote the coefficient matrix by A. The system is linear, homogeneous, with constant coefficient. We compute the eigenvalues and eigenvectors.

$$det(A-\lambda I) = det\begin{pmatrix} -1/2-\lambda & -1/8 \\ 2 & -1/2-\lambda \end{pmatrix} = (\lambda+\frac{1}{2})^2+\frac{1}{4} = 0 \Rightarrow \lambda+\frac{1}{2} = \pm\frac{i}{2}$$

Hence $\lambda = -\frac{1}{2} \pm \frac{i}{2}$.

- $\lambda = -\frac{1}{2} + \frac{i}{2}$.

$$A - \lambda I = \begin{pmatrix} -\frac{i}{2} & -1/8 \\ 2 & -\frac{i}{2} \end{pmatrix} \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ -4i \end{pmatrix}.$$

- $\lambda = -\frac{1}{2} - \frac{i}{2}$. Since A is real, the eigenvector must be conjugate of the eigenvectors we obtained in the first case. Hence

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 4i \end{pmatrix}.$$

We now have two complex solutions

$$\exp((-\frac{1}{2}+\frac{i}{2})t)\begin{pmatrix} 1 \\ -4i \end{pmatrix}, \quad \exp((-\frac{1}{2}-\frac{i}{2})t)\begin{pmatrix} 1 \\ 4i \end{pmatrix}.$$

Taking the real and imaginary parts, we have then two real solutions

$$x^{(1)} = e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\sin(t/2) \end{pmatrix}, \quad x^{(2)} = e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\cos(t/2) \end{pmatrix}.$$

We check that they are independent by doing

$$W(0) = det(x^{(1)}(0), x^{(2)}(0)) = -4 \neq 0.$$

Hence, the general solution is

$$x = C_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\sin(t/2) \end{pmatrix} + C_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\cos(t/2) \end{pmatrix}.$$

In matrix form

$$\begin{pmatrix} I \\ V \end{pmatrix} = \Psi(t)c = \begin{pmatrix} e^{-t/2}\cos(t/2) & e^{-t/2}\sin(t/2) \\ 4e^{-t/2}\sin(t/2) & -4e^{-t/2}\cos(t/2) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

(c). With the initial values, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence, $C_1 = 2, C_2 = -3/4$. Plugging these two back in, we have

$$I(t) = 2e^{-t/2}\cos(t/2) - \frac{3}{4}e^{-t/2}\sin(t/2)$$

$$V(t) = 8e^{-t/2}\sin(t/2) + 3e^{-t/2}\cos(t/2)$$

(d). Since sin, cos terms are bounded, and $\exp(-t/2) \to 0$ as $t \to \infty$, the limits must be zero for any initial condition. Physically, the system has no energy input and the energy in the circuit must be changed into heat by the resistors.