#1 (6.1-18)
$f(t) = t^n e^{at}$. According to the requirement, we evaluate

$$I_n = \int_{0^-}^{\infty} e^{-st} e^{at} t^n dt$$

Assuming $s > a$, integrating by parts, we have

$$I_n = \int_{0}^{\infty} e^{-(s-a)t} t^n dt = -\frac{1}{s-a} e^{-(s-a)t} t^n|_0^\infty + \frac{n}{s-a} \int_0^\infty e^{-(s-a)t} t^{n-1} dt$$

If $n \geq 1$, the first term is zero since $\lim_{t\to\infty} e^{-(s-a)t} t^n = 0$ and $0^n = 0$. Hence

$$I_n = \frac{n}{s-a} I_{n-1}, \quad n \geq 1$$

This implies that

$$I_n = \frac{n!}{(s-a)^n} I_0$$

We evaluate that

$$I_0 = \int_{0^-}^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a}$$

Hence the answer is

$$F(s) = I_n = \frac{n!}{(s-a)^{n+1}}$$

If we are not required to use integration by parts, we can do it as following

$$\mathcal{L}(1) = \frac{1}{s} \quad \mathcal{L}(t) = -\frac{d}{ds}(\frac{1}{s}) = \frac{1}{s^2}, \quad \ldots, \mathcal{L}(t^n) = -\frac{d}{ds}\mathcal{L}(t^{n-1}) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}t^n) = \mathcal{L}(t^n)|_{s\to s-a} = \frac{n!}{(s-a)^{n+1}}$$

#2 (6.1-31)
(a).

$$\mathcal{L}(t^p) = \int_{0^-}^{\infty} e^{-st} t^p dt$$

We do substitution $x = st$ and have $dt = dx/s$.

$$t^p = x^p/s^p$$

Hence, we have

$$\mathcal{L}(t^p) = \int_0^\infty e^{-x}\frac{x^p}{s^p}(\frac{1}{s}dx) = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p dx = \frac{\Gamma(p+1)}{s^{p+1}}$$

(b). If $p = n$, then $\Gamma(n+1) = n!$. Hence, we have

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

(c). Let $p = -1/2$, we have

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}}$$

Let's look at $\Gamma(1/2)$:

$$\Gamma(1/2) = \int_0^\infty e^{-x}x^{-1/2}dx$$

We do substitution $x = y^2$ and get

$$\Gamma(1/2) = \int_0^\infty e^{-y^2}y^{-1}(2ydy) = 2\int_0^\infty e^{-y^2}dy = \sqrt{\pi}$$

Hence, we have

$$\mathcal{L}(t^{-1/2}) = \frac{2}{s^{1/2}}\int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{s^{1/2}}$$

(d). Let $p = 1/2$. We then have

$$\mathcal{L}(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}}$$

By the identity $\Gamma(p+1) = p\Gamma(p)$, we have

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

The answer then follows.

Another way is to use the property

$$\mathcal{L}(t^{1/2}) = \mathcal{L}(t * t^{-1/2}) = -\frac{d}{ds}\mathcal{L}(t^{-1/2})$$

The same answer then follows.

#3. Let's write the right hand side as $g(t)$. We take the Laplace Transform on both sides.

$$LHS = \mathcal{L}(y'') + \mathcal{L}(y) = s^2Y(s) - sy(0) - y'(0) + Y(s) = (s^2+1)Y(s)$$

For g, we rewrite it as

$$g = t(u_0 - u_1) = tu_0 - tu_1 = tu_0 - (t-1)u_1 - u_1$$

Since $\mathcal{L}(1) = \frac{1}{s}, \mathcal{L}(t) = \frac{1}{s^2}$, we have

$$RHS = \mathcal{L}(g) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}$$

Hence

$$Y(s) = \frac{1}{s^2+1}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) = \frac{1-(1+s)e^{-s}}{s^2(s^2+1)}$$