If we are trying to find the series solutions of linear equations (which are good alternatives when we can't find the closed analytical formulas. This is usually the case when the coefficients are non-constant), we need to do the following steps:

- Pick the center of the series and get the form of the series.
- Plug in and determine the relations for the coefficients.
- If possible, determine the formula(s) for the coefficients.
- Find out n solutions in the power series form where n is the order of the equation. Verify that the radii of convergence are bigger than zero and the Wronskian is nonzero.

For the radius of convergence, if the point is a regular point, you can simply cite the theorem without applying the ratio test.

The function $y = y(x)$:

$$y'' + k^2x^2y = 0$$

The center is 0 and k is a constant.

(a). The form is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Plugging in, we have

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

For the first, $n = 0, n = 1$ are zero and hence we can safely change the index to be from $n = 2$.

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

Now, we need to make the powers agree. We can change both of them to x^n(some people used x^{n+2} and this is fine)

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} k^2 a_{n-2} x^n = 0$$

hence

$$a_2 * 2 + a_3 * 6 * x + \sum_{n=2}^{\infty} [a_{n+2}(n+2)(n+1) + k^2 a_{n-2}] x^n = 0$$

In other words, we have

$$a_2 = a_3 = 0$$
$$a_{n+2} = -\frac{k^2}{(n+2)(n+1)} a_{n-2}, n = 2, 3, \ldots$$

Or

$$a_2 = a_3 = 0$$
$$a_{n+4} = -\frac{k^2}{(n+4)(n+3)} a_n, n = 0, 1, 2, \ldots$$

(b+d): The relation is jumped by 4 and hence, we should discuss four situations: $n = 4m, 4m+1, 4m+2, 4m+3$. Since $a_2 = a_3 = 0$, it's clear that $a_{4m+2} = a_{4m+3} = 0$. Then, a_0 determines a_4, which in turn determines a_8 etc. a_1 determines a_5, and then a_9 etc. Hence, the first solution is given by $4m$ group while the second solution is given by $4m+1$ group.

We find

$$a_4 = -\frac{k^2}{4 * 3} a_0$$
$$a_8 = -\frac{k^2}{8 * 7} a_4 = (-k^2)^2 \frac{1}{(8 * 7) * (4 * 3)} a_0$$
$$\ldots$$
$$a_{4m} = (-k^2)^m \frac{1}{(4m * (4m-1)) * ((4m-4) * (4m-5)) \ldots (8 * 7) * (4 * 3)} a_0$$

Similarly, for the second group

$$a_5 = -\frac{k^2}{5 * 4} a_1$$
$$a_9 = -\frac{k^2}{9 * 8} a_5 = (-k^2)^2 \frac{1}{(9 * 8) * (5 * 4)} a_1$$
$$\ldots$$
$$a_{4m+1} = (-k^2)^m \frac{1}{((4m+1) * 4m) * ((4m-3) * (4m-4)) \ldots (9 * 8) * (5 * 4)} a_1$$

Hence,

$$y = a_0 y_1 + a_1 y_2$$

$$y_1 = 1 + \sum_{m=1}^{\infty} a_{4m} x^{4m} = 1 + \sum_{m=1}^{\infty} \frac{(-k^2 x^4)^m}{(4m*(4m-1))*((4m-4)*(4m-5))\dots(4*3)}$$

$$y_2 = x + \sum_{m=1}^{\infty} \frac{(-k^2)^m x^{4m+1}}{((4m+1)*4m)*((4m-3)*(4m-4))\dots(9*8)*(5*4)}$$

(c). Clearly, 0 is a regular point since the coefficients have Taylor series about $x_0 = 0$ and the radius of the convergence for the solution must be bigger than zero(actually it's ∞).

For the Wronskian, we compute $y_1(0) = 1, y_2(0) = 0$, $y_2'(0) = 1, y_1'(0) = 0$. Hence, we have

$$W(y_1, y_2)(0) = y_1 y_2' - y_1' y_2 = 1*1 - 0*0 = 1 \neq 0$$

$\{y_1, y_2\}$ then form a fundamental set of solutions.