#2.

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1}$$

For the first term, we define our new index to be $n = m - 2$. Then, n starts from 0. Then, the first can be written as $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$.

For the second term, we multiply x in first to get the standard form of series:

$$\sum_{k=1}^{\infty} k a_k x * x^{k-1} = \sum_{k=1}^{\infty} k a_k x^k = \sum_{n=1}^{\infty} n a_n x^n$$

Now, the two terms have the same power for x but the indices don't agree. To make them agree, we can add a term in the second series $0 * a_0 * x^0$. Notice that this term is zero and hence we don't have to subtract to balance. Therefore:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + n a_n]x^n$$

#3

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Merging the left hand side, we have

$$\sum_{n=0}^{\infty}[(n+1)a_{n+1} + 2a_n]x^n = 0$$

Hence

$$a_{n+1} = -\frac{2}{n+1}a_n, \qquad n = 0, 1, 2, 3, \ldots$$

Using this relation, we find a formula for a_n:

$$a_1 = -\frac{2}{1}a_0$$
$$a_2 = -\frac{2}{2}a_1$$
$$a_3 = -\frac{2}{3}a_2$$
$$\cdots$$
$$a_n = -\frac{2}{n}a_{n-1}$$

Hence, $a_n = (-2)^n \frac{1}{n*(n-1)*...*2*1} a_0 = \frac{(-2)^n}{n!} a_0$.

Hence,

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = a_0 e^{-2x}$$

Actually, the equation is $y' + 2y = 0$. It's natural to get $y = Ce^{-2x}$