1 Hw 4

#1(c).

The characteristic equation is \(9r^2 - 12r + 4 = 0\) or \((3r - 2)^2 = 0\). Hence, there is one repeated root \(r = 2/3\). A fundamental set is \(\{e^{2t/3}, te^{2t/3}\}\) and the general solution is

\[
y = C_1 \exp\left(\frac{2t}{3}\right) + C_2 t \exp\left(\frac{2t}{3}\right)
\]

Using the initial conditions

\[
C_1 + 0 = 2 \\
\frac{2}{3}C_1 + C_2 = -2
\]

Hence \(C_1 = 2, C_2 = -\frac{10}{3}\). The solution to the IVP is

\[
y = 2e^{2t/3} - \frac{10}{3}te^{2t/3}
\]

#2. The main issue is that some people were using the characteristic equation for this problem. That is wrong! This is not a constant-coefficient equation and the characteristic equation doesn’t work.

To check, you just plug in. To see if they make a fundamental set, you just compute the Wronskian. I would like to omit the checking process. For the Wronskian, \(W(y_1, y_2) = y_1y_2' - y_2y_1' = (e^t + te^t) - te^t = t^2e^t\). This is nonzero for all \(t > 0\). Hence, they make a fundamental set.

2 The suggested problems

#1. The homogeneous equation is that \(y'' + 2y' + 5y = 0\) and the characteristic equation is \(r^2 + 2r + 5 = 0\). Hence, \(r = -1 \pm 2i\). \(\{e^{-t}\cos(2t), e^{-t}\sin(2t)\}\).

The forcing term is just a solution to the homogeneous equation. The particular solution for this problem can be done by the undetermined coefficients by trying \(y_p = C_1te^{-t}\cos(2t) + C_2te^{-t}\sin(2t)\). Note that we don’t try \(y_p = C_1te^{-t}\cos(2t) + C_2te^{-t}\sin(2t) + C_3e^{-t}\cos(2t) + C_4e^{-t}\sin(2t)\) because the latter two are solutions to the homogeneous equation.

Now let’s plug in. Denote the differential operator on left by \(L\).

\[
L(C_1te^{-t}\cos(2t) + C_2te^{-t}\sin(2t)) = C_1L(te^{-t}\cos(2t)) + C_2L(te^{-t}\sin(2t))
\]
By Leibnitz rule

\[(te^{-t} \cos(2t))'' = 0 + 2 \cdot 1 \cdot (e^{-t} \cos(2t))' + t(e^{-t} \cos(2t))''\]

and the product rule

\[(te^{-t} \cos(2t))' = e^{-t} \cos(2t) + t(e^{-t} \cos(2t))'\]

and using the expression of \(L\), we find that

\[L(te^{-t} \cos(2t)) = 2 \cdot 1 \cdot (e^{-t} \cos(2t))' + 2 \cdot e^{-t} \cos(2t) + t \cdot L(e^{-t} \cos(2t))\]

The last term is zero since \(e^{-t} \cos(2t)\) solves the homogeneous equation. Similarly

\[L(te^{-t} \sin(2t)) = 2(e^{-t} \sin(2t))' + 2e^{-t} \sin(2t) + t \cdot L(e^{-t} \sin(2t))\]

Hence, the left hand side is

\[C_1(2 \cdot 1 \cdot (e^{-t} \cos(2t))' + 2 \cdot e^{-t} \cos(2t)) + C_2(2(e^{-t} \sin(2t))' + 2e^{-t} \sin(2t))\]

\[= -4C_1 e^{-t} \sin(2t) + 4C_2 e^{-t} \cos(2t)\]

We see then that \(C_1 = 0, C_2 = 1\). Hence,

\[Y = te^{-t} \sin(2t)\]

The general solution is

\[y = A_1 e^{-t} \cos(2t) + A_2 e^{-t} \sin(2t) + te^{-t} \sin(2t)\]

\[y(0) = A_1 = 1\]

Then,

\[y'(0) = -A_1 + 2A_2 + 0 = 0, A_2 = 1/2\]

#2. Once again, the homogeneous solution can’t be solved using the characteristic equation because it’s not constant coefficient.

(b). The verification is easy. We just plug in. For example, for \(y_2 = t^2 \ln t\), by plugging in,

\[LHS = t^2(t^2 \ln t)^'' - 3t(t^2 \ln t)' + 4(t^2 \ln t) = t^2(2 \ln t + 4 - 1) - 3t(2 \ln t + t) + 4t^2 \ln t = 0 = RHS\]
Let’s find \( y_p \) by the variation of parameters. One mistake people tend to make is to use \( g = t^2 \ln t \). This is not OK. We first should normalize the coefficient of \( y'' \) to 1. The equations

\[
\begin{align*}
u_1'y_1 + u_2'y_2 &= 0 \\
u_1'y_1' + u_2'y_2' &= t^2 \ln t/t^2
\end{align*}
\]

we can compute \( W(y_1, y_2) = t^2(2t \ln t + t) - 2t^2 \ln t = t^3 \). We solve that

\[
\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \ln t \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ \ln t \end{bmatrix}
\]

Hence, \( u_1' = -\frac{y_2 \ln t}{W} = -(\ln t)^2/t \), which yields \( u_1 = -\int \frac{1}{t}(\ln t)^2 dt = -\frac{1}{3}(\ln t)^3 \) (we only need one and hence the constant is ignored).

Similarly, \( u_2' = \frac{1}{t} \ln t \) and thus \( u_2 = \frac{1}{2}(\ln t)^2 \).

\[
y_p = t^2\left(-\frac{1}{3}(\ln t)^3\right) + t^2 \ln t \left(\frac{1}{2}(\ln t)^2\right) = \frac{1}{6} t^2 (\ln t)^3
\]