1 Hw 4

#1(c).

The characteristic equation is $9r^2-12r+4=0$ or $(3r-2)^2=0$. Hence, there is one repeated root $r=2/3$. A fundamental set is $\{e^{2t/3}, te^{2t/3}\}$ and the general solution is

$$y = C_1\exp(\frac{2t}{3}) + C_2 t\exp(\frac{2t}{3})$$

Using the initial conditions

$$C_1 + 0 = 2$$
$$\frac{2}{3}C_1 + C_2 = -2$$

Hence $C_1 = 2, C_2 = -\frac{10}{3}$. The solution to the IVP is

$$y = 2e^{2t/3} - \frac{10}{3}te^{2t/3}$$

#2. The main issue is that some people were using the characteristic equation for this problem. That is wrong! This is not a constant-coefficient equation and the characteristic equation doesn't work.

To check, you just plug in. To see if they make a fundamental set, you just compute the Wronskian. I would like to omit the checking process. For the Wronskian, $W(y_1,y_2) = y_1y_2' - y_2y_1' = t(e^t+te^t) - te^t = t^2e^t$. This is nonzero for all $t>0$. Hence, they make a fundamental set.

2 The suggested problems

#1. The homogeneous equation is that $y''+2y'+5y=0$ and the characteristic equation is $r^2+2r+5=0$. Hence, $r=-1\pm 2i$. $\{e^{-t}\cos(2t), e^{-t}\sin(2t)\}$.

The forcing term is just a solution to the homogeneous equation. The particular solution for this problem can be done by the undetermined coefficients by trying $y_p = C_1te^{-t}\cos(2t)+C_2te^{-t}\sin(2t)$. Note that we don't try $y_p = C_1te^{-t}\cos(2t)+C_2te^{-t}\sin(2t)+C_3e^{-t}\cos(2t)+C_4e^{-t}\sin(2t)$ because the latter two are solutions to the homogeneous equation.

Now let's plug in. Denote the differential operator on left by L.

$$L(C_1te^{-t}\cos(2t)+C_2te^{-t}\sin(2t)) = C_1L(te^{-t}\cos(2t)) + C_2L(te^{-t}\sin(2t))$$

By Leibnitz rule

$$(te^{-t}\cos(2t))'' = 0 + 2*1*(e^{-t}\cos(2t))' + t(e^{-t}\cos(2t))''$$

and the product rule

$$(te^{-t}\cos(2t))' = e^{-t}\cos(2t) + t(e^{-t}\cos(2t))'$$

and using the expression of L, we find that

$$L(te^{-t}\cos(2t)) = 2*1*(e^{-t}\cos(2t))' + 2*e^{-t}\cos(2t) + t*L(e^{-t}\cos(2t))$$

The last term is zero since $e^{-t}\cos(2t)$ solves the homogeneous equation. Similarly

$$L(te^{-t}\sin(2t)) = 2(e^{-t}\sin(2t))' + 2e^{-t}\sin(2t) + t*L(e^{-t}\sin(2t))$$

Hence, the left hand side is

$$C_1(2*1*(e^{-t}\cos(2t))' + 2*e^{-t}\cos(2t)) + C_2(2(e^{-t}\sin(2t))' + 2e^{-t}\sin(2t))$$
$$= -4C_1e^{-t}\sin(2t) + 4C_2e^{-t}\cos(2t)$$

We see then that $C_1 = 0, C_2 = 1$. Hence,

$$Y = te^{-t}\sin(2t)$$

The general solution is

$$y = A_1e^{-t}\cos(2t) + A_2e^{-t}\sin(2t) + te^{-t}\sin(2t)$$

$$y(0) = A_1 = 1$$

Then,

$$y'(0) = -A_1 + 2A_2 + 0 = 0, A_2 = 1/2$$

#2. Once again, the homogeneous solution can't be solved using the characteristic equation because it's not constant coefficient.

(b). The verification is easy. We just plug in. For example, for $y_2 = t^2\ln t$, by plugging in,

$$LHS = t^2(t^2\ln t)'' - 3t(t^2\ln t)' + 4(t^2\ln t) = t^2(2\ln t + 4 - 1) - 3t(2t\ln t + t) + 4t^2\ln t = 0 = RHS$$

Let's find y_p by the variation of parameters. One mistake people tend to make is to use $g = t^2 \ln t$. This is not OK. We first should normalize the coefficient of y'' to 1 The equations

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = t^2 \ln t / t^2$$

we can compute $W(y_1, y_2) = t^2(2t \ln t + t) - 2tt^2 \ln t = t^3$. We solve that

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \ln t \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ \ln t \end{bmatrix}$$

Hence, $u_1' = \frac{-y_2 \ln t}{W} = -(\ln t)^2/t$, which yields $u_1 = -\int \frac{1}{t}(\ln t)^2 dt = -\frac{1}{3}(\ln t)^3$(we only need one and hence the constant is ignored).

Similarly, $u_2' = \frac{1}{t} \ln t$ and thus $u_2 = \frac{1}{2}(\ln t)^2$.

$$y_p = t^2(-\frac{1}{3}(\ln t)^3) + t^2 \ln t(\frac{1}{2}(\ln t)^2) = \frac{1}{6} t^2 (\ln t)^3$$