#1.

(a). To verify they are the solutions for the IVP, we need two things: They solve the differential equation on a certain interval; They satisfy the initial conditions.

For $y_1(t) = 1 - t$. The initial condition is obvious, because $y_1(2) = 1 - 2 = -1$. For the differential equation, we just plug in

$$f(t, y_1) = \frac{-t + \sqrt{t^2 + 4(1-t)}}{2} = \frac{-t + |t-2|}{2} = \begin{cases} -1 & t \geq 2 \\ 1-t & t < 2 \end{cases}$$

We also verify that $y_1'(t) = -1$. Hence, y_1 solves the ODE on $t \geq 2$.

For $y_2(t)$, the initial condition is $y_2(2) = -\frac{2^2}{4} = -1$. We also check that

$$f(t, y_2) = \frac{-t + \sqrt{t^2 + 4(-t^2/4)}}{2} = -\frac{t}{2}$$

and $y_2'(t) = -t/2$. Hence, y_2 also solves the IVP.

Some people were only checking the ODE at $t = 2$. This is not good. We should check the equation for any t on a certain interval.

The interval of definition is the one that includes t_0 and on which a function y is defined and solves the equation. The boundary is where the function begins to behave poorly (i.e. either y or y' blows up) or the ODE begins to fail.

In our case, we see that y and y' are defined everywhere. However, for y_1, the ODE fails to be true on $t < 2$. Hence the interval of definition for y_1 is $[2, \infty)$ while the interval for y_2 is the whole $\mathbb{R}$

(b). There's no contradiction. One requirement for the the theorem is that both f and f_y are continuous. However, in our example, $f_y = \frac{1}{\sqrt{t^2+4y}}$ is not continuous at $(t, y) = (2, -1)$

(c). To verify it's a solution, it's easy. We just plug in and see $f(t, y) = \frac{-t+|t+2c|}{2}$ and $y' = c$. Hence, if $t \geq -2c$, $f(t, y) = c$ and the equation holds. That means y is a solution for $t \geq -2c$. When, $c = -1$, it's just y_1 and it satisfies the initial condition.

We see that $y = ct + c^2$ is linear function while y_2 is a quadratic function. These two can't be equal since $-t^2/4 - ct - c^2$ is not always zero. Hence, y_2 can't be included into this family. This means generally there is no general solution (or a formula that contains all solutions) for a nonlinear equation.