9.6#3

The Lyapunov function can be constructed to be positive definite. Hence, we need $a > 0, c > 0$. Then, along a trajectory $x(t), y(t)$, we have (All x, y should be understood as $x(t), y(t)$):

$$\frac{d}{dt}V = V_x\dot{x} + V_y\dot{y} = 2ax(-x^3 + 2y^3) + 2cy(-2xy^2) = -2ax^4 + 4(a-c)xy^3$$

The first term is nonpositive but the second term is bad since the sign can't be determined. Hence, we can conveniently choose $a = c = 1$ to kill it.

$V = x^2 + y^2$ is clearly positive definite and $\frac{d}{dt}V = -2x^4 \leq 0$ which is negative semidefinite. Hence, the critical point $(0, 0)$ is at least stable.

9.6.#9

(a). If we do substitution $x = u, y = du/dt$, we have

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -y - g(x) \end{cases}$$

Clearly, $(0, 0)$ is a critical point.

If we use the Lyapunov function

$$V = \frac{1}{2}y^2 + \int_0^x g(s)ds,$$

we first of all should show that V is positive definite in a neighborhood of the critical point $(0, 0)$ and show that $\frac{d}{dt}V(x(t), (t))$ is negative semidefinite.

The condition $V(0, 0) = 0$ is clear. For a point (x, y) such that $0 < |x| < k$, we have $\int_0^x g(s)ds > 0$. This is because when $x > 0$, $g(s) > 0$ on $(0, x]$; when $x < 0$, $\int_0^x g(s)ds = -\int_0^{-x} g(-s)ds > 0$ as $\int_0^{-x} g(-s)ds < 0$ since $g(-s) < 0$. Hence, $V(x, y) = \frac{1}{2}y^2 + \int_0^x g(s)ds > 0$ for any $(x, y) \neq (0, 0)$ and $|x| < k$. V is positive definite in $(-k, k) \times (-k, k)$.

Now, along any trajectory $(x(t), y(t))$, we compute

$$\frac{d}{dt}V(x(t), y(t)) = V_x\dot{x} + V_y\dot{y} = g(x) * y + y * (-y - g(x)) = -y^2 \leq 0$$

This then verifies that the critical point $(0, 0)$ is stable.

(b). In this problem, we show the asymptotical stability for the special case $g(x) = \sin(x)$ as the problem required. (For general g, it involves some Taylor expansion estimates.)

As the problem suggests, we use the following Lyapunov function

$$V(x, y) = \frac{1}{2}y^2 + \frac{1}{2}y\sin(x) + \int_0^x \sin(s)ds = \frac{1}{2}y^2 + \frac{1}{2}y\sin x + 1 - \cos(x)$$

We now show that this is positive definite. Using the expansion $\sin x = x - \alpha x^3/3!$ and $\cos(x) = 1 - \frac{x^2}{2} + \gamma\frac{x^4}{4!}$, we have

$$V = \frac{1}{2}y^2 + \frac{1}{2}xy + \frac{x^2}{2} - \alpha\frac{x^3 y}{2 * 3!} - \gamma\frac{x^4}{4!}$$

We note that $|\alpha| < 1, |\gamma| < 1$ and $x^3 y = o(x^2 + y^2), x^4 = o(x^2 + y^2)$. For the latter two, we have

$$|\frac{x^3 y}{x^2 + y^2}| \leq |xy| \to 0, |\frac{x^4}{x^2 + y^2}| \leq |x^2| \to 0.$$

Hence, for any $\epsilon > 0$, we can find $\delta > 0$ such that when $r = \sqrt{x^2 + y^2} < \delta$, we have

$$-|\alpha\frac{x^3 y}{2 * 3!} + \gamma\frac{x^4}{4!}| \geq -\epsilon(x^2 + y^2)$$

Then,

$$V \geq \frac{1}{2}y^2 + \frac{1}{2}xy + \frac{1}{2}x^2 - \epsilon(x^2 + y^2) = (\frac{1}{2} - \epsilon)y^2 + \frac{1}{2}xy + (\frac{1}{2} - \epsilon)x^2$$

we see that as long as $\epsilon < 1/4$, $4AC - B^2 > 0$ and the Lyapunov function is positive definite.

For the derivative:

$$\frac{d}{dt}V = V_x\dot{x} + V_y\dot{y} = (\frac{y}{2}\cos x + \sin(x))y + (y + \frac{\sin x}{2})(-y - \sin(x))$$
$$= (\frac{1}{2}\cos x - 1)y^2 - \frac{1}{2}y\sin x - \frac{1}{2}\sin^2 x = (\frac{1}{2}\cos x - 1)y^2 - \frac{1}{2}y\sin x - \frac{1}{2}\sin^2 x$$

Again, we use the expansion:

$$\dot{V} = -\frac{1}{2}y^2 - \frac{\beta x^2 y^2}{4} - \frac{1}{2}xy + \alpha\frac{1}{2 * 3!}x^3 y - \frac{1}{2}x^2 + \frac{2\alpha x^4}{3!} - \frac{\alpha^2 x^6}{(3!)^2}$$

Similarly, x^2y^2, x^3y, x^4, x^6 are all $o(x^2 + y^2)$(that means they decay to zero faster than $x^2 + y^2$). Then, for any $\epsilon > 0$, we can find $\delta > 0$ such that if $r < \delta$, we have

$$\dot{V} \leq -\frac{1}{2}y^2 - \frac{1}{2}xy - \frac{1}{2}x^2 + \epsilon(x^2 + y^2)$$

If we choose $0 < \epsilon < 1/4$, then, $\dot{V}$ is negative definite. The critical point $(0, 0)$ is asymptotically stable.

Another simpler Lyapunov function for this problem is

$$V(x,y) = \frac{1}{2}y^2 + \frac{1}{2}xy + \int_0^x \sin(s)ds = \frac{1}{2}y^2 + \frac{1}{2}xy + 1 - \cos(x)$$
$$= \frac{1}{2}(y + x/2)^2 + 1 - \cos x - x^2/8$$

We need to show that this is positive definite. We need the expansion of $\cos(x)$:

$$\cos(x) = 1 - \frac{x^2}{2!} + \gamma\frac{x^4}{4!}$$

Hence, we have

$$1 - \cos x - \frac{x^2}{8} = \frac{x^2}{2} - \frac{x^2}{8} - \gamma\frac{x^4}{4!} = \frac{3}{8}x^2 - \gamma\frac{x^4}{4!}$$

Hence, we can find numbers $\delta > 0$ and $\mu \in (0, 3/8)$ such that when $|x| < \delta$,

$$\frac{3}{8}x^2 - \gamma\frac{x^4}{4!} \geq \mu x^2$$

since x^4 decays to zero much faster than x^2 as $x \to 0$.
Then V is positive definite in $|x| < \delta, |y| < \delta$ since $\frac{1}{2}(y + x/2)^2 + \mu x^2 > 0$ for $(x, y) \neq (0, 0)$.

The derivative can be computed:

$$\frac{d}{dt}V = V_x\dot{x} + V_y\dot{y} = (\frac{y}{2} + \sin(x))y + (y + \frac{x}{2})(-y - \sin(x))$$
$$= -\frac{y^2}{2} - \frac{xy}{2} - \frac{x\sin(x)}{2} = -\frac{1}{2}(y + x/2)^2 - \frac{x\sin x}{2} + \frac{x^2}{8}$$

This time, we use the expansion of $\sin x$: $\sin x = x - \alpha x^3/3!$. We find that

$$-\frac{x\sin x}{2} + \frac{x^2}{8} = -\frac{3}{8}x^2 + \alpha\frac{x^4}{2 * 3!}.$$

Similarly, we can find $\delta > 0$ and $\mu \in (0, 3/8)$ such that

$$-\frac{3}{8}x^2 + \alpha\frac{x^4}{2 * 3!} \leq -\mu x^2$$

since x^4 decays to zero much faster than x^2 as $x \to 0$.
Then, $\frac{d}{dt}V$ is negative definite. The critical point $(0, 0)$ is asymptotically stable.

#10. (a). We first show Sec. 9.1, 21:

Let's denote $p = a_{11} + a_{22}$ for the trace and $q = a_{11}a_{22} - a_{12}a_{21}$ for the determinant. Then, the eigenvalues satisfy

$$\lambda^2 - p\lambda + q = 0$$

(c). In the case $p > 0$, $\lambda_1 + \lambda_2 = p > 0$, then one of them must have a positive real part. Hence, unstable. In the case $q < 0$, $\lambda_1\lambda_2 = q < 0$, then since the matrix is real, either both λ_1 and λ_2 are real or they are conjugate to each other. The latter can't happen since the product of them is negative. Hence, both of them are real. The negative product implies that one of them is positive and hence the critical point is unstable.

(b). In the case $q > 0$ and $p = 0$, the eigenvalues are pure imaginary. The critical point of the linear system is stable but not asymptotically stable.

(a). In the case, $q > 0, p < 0$, we have

$$\lambda = \frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

Since $q > 0$, $|\sqrt{p^2 - 4q}| < -p$ and the real parts of the eigenvalues must be negative no matter if $p^2 - 4q$ is positive or negative. Then, the critical point is asymptotically stable.

Note, you can't say 'Because asymptotically stable, by 9.1, 21(a), $q > 0, p < 0$' since we haven't shown the reverse direction.

We show that the implications in Sec. 9.1, 21 (a) can be reversed. If we have the asymptotically stability but $q > 0, p < 0$ is not true, then one of the three cases '$q > 0, p = 0$', '$q < 0$','$p > 0$' must happen. However, as we have shown in 9.1, 21(b) or 21(c), any of them will not give the asymptotical stability. Hence, $q > 0, p < 0$ must be true.

(b). As indicated, we first choose A, B, C to make $\dot{V} = -x^2 - y^2$ and then show that V is positive definite in (c).

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = (2Ax + By)(a_{11}x + a_{12}y) + (Bx + 2Cy)(a_{21}x + a_{22}y)$$

To satisfy the requirements, we have

$$2Aa_{11} + Ba_{21} = -1$$
$$2Aa_{12} + Ba_{11} + Ba_{22} + 2Ca_{21} = 0$$
$$Ba_{12} + 2Ca_{22} = -1$$

We have the linear system

$$\begin{bmatrix} 2a_{11} & a_{21} & 0 \\ 2a_{12} & a_{11}+a_{22} & 2a_{21} \\ 0 & a_{12} & 2a_{22} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

We denote the coefficient matrix by M. To solve the unknowns numerically, usually we use Gauss elimination. However, here, we prefer to use Cramer's rule to derive the formulas. By Cramer's rule, we have

$$A = \frac{1}{det(M)} det \begin{bmatrix} -1 & a_{21} & 0 \\ 0 & a_{11}+a_{22} & 2a_{21} \\ -1 & a_{12} & 2a_{22} \end{bmatrix} = \frac{-2(a_{11}a_{22}+a_{22}^2-a_{12}a_{21}+a_{21}^2)}{4\Delta}$$

$$B = \frac{1}{detM} det \begin{bmatrix} 2a_{11} & -1 & 0 \\ 2a_{12} & 0 & 2a_{21} \\ 0 & -1 & 2a_{22} \end{bmatrix} = \frac{4(a_{12}a_{22}+a_{11}a_{21})}{4\Delta}$$

Similarly,

$$C = \frac{1}{det(M)} det \begin{bmatrix} 2a_{11} & a_{21} & -1 \\ 2a_{12} & a_{11}+a_{22} & 0 \\ 0 & a_{12} & -1 \end{bmatrix} = \frac{-2(a_{11}^2+a_{12}^2+a_{11}a_{22}-a_{12}a_{21})}{4\Delta}$$

(c). Clearly, $\Delta = pq < 0$. We see that $q = a_{11}a_{22} - a_{12}a_{21} > 0$ and hence $A = -(q + a_{21}^2 + a_{22}^2)/(2\Delta) > 0$.

Clearly,

$$4AC - B^2 = \frac{(a_{11}a_{22}+a_{22}^2-a_{12}a_{21}+a_{21}^2)(a_{11}^2+a_{12}^2+a_{11}a_{22}-a_{12}a_{21}) - (a_{12}a_{22}+a_{11}a_{21})^2}{\Delta^2}$$

The numerator can be computed as

$$\begin{aligned} &(q + a_{21}^2 + a_{22}^2)(q + a_{11}^2 + a_{12}^2) - (a_{12}a_{22} + a_{11}a_{21})^2 \\ &= q^2 + (a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)q + (a_{21}^2a_{12}^2 + a_{22}^2a_{11}^2 - 2a_{12}a_{22}a_{11}a_{21}) \\ &= 2q^2 + (a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)q \end{aligned}$$

Since $q > 0$, $4AC - B^2 > 0$, the Lyapunov function is positive definite.

#11. (a). Using the same Lyapunov function

$$\begin{aligned} \dot{V} = V_x\dot{x} + V_y\dot{y} &= (2Ax + By)(a_{11}x + a_{12}y + F_1) + (Bx + 2Cy)(a_{21}x + a_{22}y + G_1) \\ &= -x^2 - y^2 + (2Ax + By)F_1 + (Bx + 2Cy)G_1. \end{aligned}$$

(b). Since $F_1 = o(r), G_1 = o(r)$(in other words, $F_1/r \to 0, G_1/r \to 0$ as $r \to 0$), given any $\epsilon > 0$, we can choose $\delta > 0$ such that

$$|F_1| < \epsilon r, |G_1| < \epsilon r, if\ 0 < r < \delta$$

Then,

$$\begin{aligned}|(2Ax + By)F_1 + (Bx + 2Cy)G_1| &\le |2A||x|\epsilon r + |B||y|\epsilon r + |B||x|\epsilon r + |2C||y|\epsilon r \\ &\le 2(|A| + |B| + |C|)\epsilon r^2\end{aligned}$$

If we choose $\epsilon = 1/(4(|A| + |B| + |C|))$, then this term is $\le r^2/2$. Hence we have

$$\dot{V} \le -x^2 - y^2 + \frac{r^2}{2} = -\frac{x^2 + y^2}{2}$$

which shows that the derivative is negative definite.

Hence, the nonlinear system is also asymptotically stable at $(0, 0)$.