9.1#4
(a).

$$det(A - \lambda I) = (1-\lambda)(-7-\lambda) + 16 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0$$

Hence the eigenvalue is $\lambda = -3$, which is repeated.

$$A - (-3)I = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}.$$

Hence, there is only one independent eigenvector since there is only one free variable in the system

$$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b). There is a repeated eigenvalue with one independent eigenvector, the critical point is an improper node. Since $Re(\lambda) < 0$ for both eigenvalues(though repeated), every solution tends to zero, which means that any error to the critical point $(0,0)$ (which is a constant solution) will eventually vanish. Hence, the critical point is asymptotically stable(and of course stable).

(c). One solution is constructed by the eigenvector

$$x^{(1)} = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The other solution is found by the generalized eigenvector

$$[A - (-3)I]\eta = \xi \Rightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

One possible η is found:

$$\eta = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$$

The second solution is

$$x^{(2)} = te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}.$$

All solutions can be written as

$$x = C_1 x^{(1)} + C_2 x^{(2)} = (C_1 + C_2 t)e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}.$$

Some typical trajectories are shown in Fig. 1:

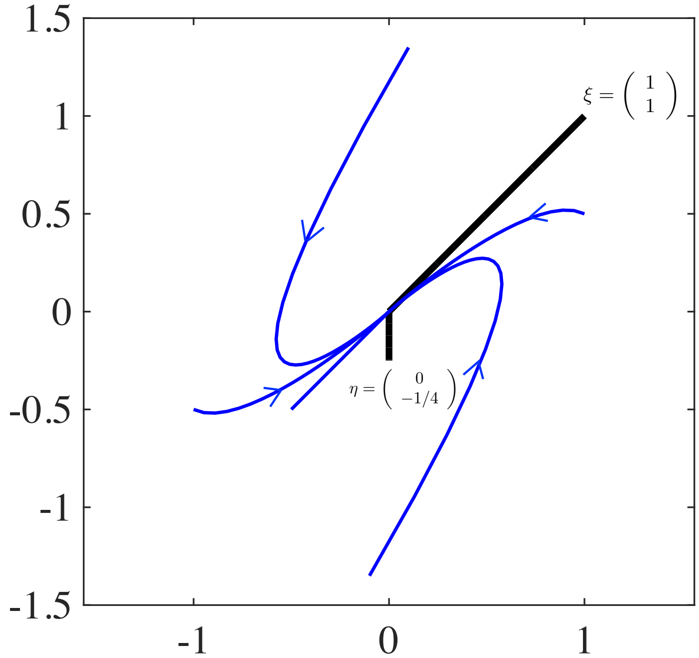

Figure 1: Trajectories

It's easy to see that the function x_1 is

$$x_1(t) = (C_1 + C_2 t)e^{-3t}$$

You can plot some typical graphs. This is the easy part and I would like to omit.

9.3#30.

(a). Introducing $y = dx/dt$, we have

$$\begin{cases} x' = y \\ y' = x'' = -c(x)x' - g(x) = -c(x)y - g(x) \end{cases}$$

Hence

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -c(x)y - g(x) \end{pmatrix} =: \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}.$$

where =: means 'the right hand side is defined by the left hand side'.

(b). We just plug in $F(0,0) = y|_{(x,y)=(0,0)} = 0$ and $G(0,0) = -c(0) * 0 - g(0) = -g(0) = 0$. Hence, it is a critical point.

Then, since both F and G are continuously differentiable at $(0,0)$, we have by Taylor expansion

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o(\sqrt{x^2+y^2}) = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c(0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o(\sqrt{x^2+y^2})$$

The Jacobian matrix is nonzero, and hence the system is locally linear in the neighborhood of the critical point.

(c). We determine the stability of the linear system in (b). We compute the characteristic equation

$$\lambda(\lambda + c(0)) + g'(0) = 0 \Rightarrow \lambda = \frac{-c(0) \pm \sqrt{c^2(0) - 4g'(0)}}{2}$$

In the case that $c(0) > 0, g'(0) > 0$, $\sqrt{c^2(0) - 4g'(0)}$ is either pure imaginary or a real number which is smaller than $c(0)$. Hence, $Re(\lambda_+) < 0$ and $Re(\lambda_-) < 0$. The linear system is asymptotically stable. Since the real part of the eigenvalue is nonzero, the stability of the nonlinear system is the same as the linear system. The conclusion follows.

In the case that $c(0) < 0$ and $g'(0) < 0$, $\sqrt{c^2(0) - 4g'(0)} > |c(0)|$. Hence, we must have $Re(\lambda_+) > 0$ which implies the linear system is unstable. Again, the real part is nonzero, and the stability of the nonlinear system is the same as the linear system. The conclusion follows.