#4.

The equation is separable. Before dividing $y(4-y)$, we check if there are constant solutions to make it zero.

$$y = 0, y = 4$$

Now, assume $y \neq 0, 4$.

$$\int \frac{1}{y(4-y)} dy = \int \frac{t}{t+1} dy$$

$$LHS = \frac{1}{4} \int (\frac{1}{y} + \frac{1}{4-y}) dy = \frac{1}{4} \ln |\frac{y}{4-y}|$$

where the constant is absorbed to RHS.

$$RHS = \int (1 - \frac{1}{t+1}) dt = t - \ln |1+t| + C_1$$

Hence,

$$\frac{y}{4-y} = C(1+t)^{-4} e^{4t} \Rightarrow y = \frac{4C(1+t)^{-4}e^{4t}}{1 + C(1+t)^{-4}e^{4t}}$$

(a). With $y(0) = y_0$, we see that if $y_0 = 4$, $y = 4$. If $y_0 > 0, y_0 \neq 4$, $C = \frac{y_0}{4-y_0}$. Hence the solution is

$$y = \frac{4y_0 e^{4t}}{(4-y_0)(1+t)^4 + y_0 e^{4t}}$$

Interestingly, even though $y = 4$ can't be combined into the expression that contains C, this reduced solution is valid for all $y_0 > 0$ including $y_0 = 4$.

For $y_0 > 0$, taking the limit $t \to \infty$, since $e^{-4t}(1+t)^4 \to 0$, the limit is $\frac{4y_0}{y_0} = 4$.

Some people were analyzing the behavior by the equation directly. That's great but the argument was not convincing. Let me provide one here:

It's clear that $y = 0, y = 4$ are two solutions. Also, if $y_0 > 4$, $y > 4$ always because otherwise $y(t_0) = 4$ for some $t_0 > 0$. Then, around t_0, there are two solutions of the equation that satisfies $y(t_0) = 4$ which contradicts with the existence and uniqueness theorem. Similarly if $0 < y_0 < 4$, $0 < y < 4$ is true always. If $y_0 = 4$, $y = 4$. This means the constant solutions will control the non constant solutions.

Now, if $y_0 > 4$ and hence $y > 4$, we see that $y'(t) < 0$ and thus y decreases. Because it's bounded below by 4, $\lim_{t\to\infty} y = Y$ exists. Also,

$y'(t) \to 0, t \to \infty$ if $y \to Y$ because the function is differentiable for as many times as you want. (To be rigorous, $\int_T^{T+\Delta} y'(t)dt = y(T+\Delta) - y(T) \to Y - Y = 0$. However, $y' < 0$, $-\int_T^{T+\Delta} |y'|dt \to 0$. However, by the equation, $|y'| \to const$. Then, $|y'| \to 0$). You take the limit on both sides of the equation and have $0 = Y(4-Y)$. Since $y > 4$, the only possibility is $Y = 4$.

If $0 < y_0 < 4$, the solution increases but will be always below 4. Similarly the limit exists. Also, $y' \to 0$ if the limit exists. Then, the same trick applies. The limit is 4 again.

(b). When $y_0 = 2$, the solution is $y(t) = \frac{4e^{4t}}{(1+t)^4 + e^{4t}}$. Hence, if $y(T) = 3.99$,

$$e^{4T} = 399(1+T)^4$$

Here we seek the time that is on the right of $t = 0$ and it is $T \approx 2.8437$(the $-0.91, -1.07$ are discarded because they are not the 'first time' as we increase time from $t = 0$)

(c). We compute $y(2) = \frac{4y_0e^8}{3^4(4-y_0)+y_0e^8} \approx \frac{4y_0}{0.0272(4-y_0)+y_0} = \frac{4y_0}{0.9728y_0+0.1087}$

When this is 4.01, $y_0 \approx 4.40$. When this is 3.99, $y_0 \approx 3.66$. Hence $3.66 < y_0 < 4.40$