#4.

The equation is separable. Before dividing \( y(4 - y) \), we check if there are constant solutions to make it zero.

\[ y = 0, y = 4 \]

Now, assume \( y \neq 0, 4 \).

\[
\int \frac{1}{y(4 - y)}dy = \int \frac{t}{t + 1}dy
\]

\[ \text{LHS} = \frac{1}{4} \int \left( \frac{1}{y} + \frac{1}{4 - y} \right)dy = \frac{1}{4} \ln \left| \frac{y}{4 - y} \right| \]

where the constant is absorbed to RHS.

\[ \text{RHS} = \int (1 - \frac{1}{t + 1})dt = t - \ln |1 + t| + C_1 \]

Hence,

\[
\frac{y}{4 - y} = C(1 + t)^{-4}e^{4t} \Rightarrow y = \frac{4C(1 + t)^{-4}e^{4t}}{1 + C(1 + t)^{-4}e^{4t}}
\]

(a). With \( y(0) = y_0 \), we see that if \( y_0 = 4, y = 4 \). If \( y_0 > 0, y_0 \neq 4 \), \( C = \frac{y_0}{4 - y_0} \). Hence the solution is

\[
y = \frac{4y_0e^{4t}}{(4 - y_0)(1 + t)^4 + y_0e^{4t}}
\]

Interestingly, even though \( y = 4 \) can’t be combined into the expression that contains \( C \), this reduced solution is valid for all \( y_0 > 0 \) including \( y_0 = 4 \).

For \( y_0 > 0 \), taking the limit \( t \to \infty \), since \( e^{-4t}(1 + t)^4 \to 0 \), the limit is \( \frac{4y_0}{y_0} = 4 \).

Some people were analyzing the behavior by the equation directly. That’s great but the argument was not convincing. Let me provide one here:

It’s clear that \( y = 0, y = 4 \) are two solutions. Also, if \( y_0 > 4 \), \( y > 4 \) always because otherwise \( y(t_0) = 4 \) for some \( t_0 > 0 \). Then, around \( t_0 \), there are two solutions of the equation that satisfies \( y(t_0) = 4 \) which contradicts with the existence and uniqueness theorem. Similarly if \( 0 < y_0 < 4, \ 0 < \ y < 4 \) is true always. If \( y_0 = 4, y = 4 \). This means the constant solutions will control the non constant solutions.

Now, if \( y_0 > 4 \) and hence \( y > 4 \), we see that \( y'(t) < 0 \) and thus \( y \) decreases. Because it’s bounded below by 4, \( \lim_{t \to \infty} y = Y \) exists. Also,
$y'(t) \to 0, t \to \infty$ if $y \to Y$ because the function is differentiable for as many times as you want. (To be rigorous, $\int_{T}^{T+\Delta} y'(t) dt = y(T+\Delta) - y(T) \to Y - Y = 0$. However, $y' < 0, -\int_{T}^{T+\Delta} |y'| dt \to 0$. However, by the equation, $|y'| \to \text{const}$. Then, $|y'| \to 0$). You take the limit on both sides of the equation and have $0 = Y(4-Y)$. Since $y > 4$, the only possibility is $Y = 4$.

If $0 < y_0 < 4$, the solution increases but will be always below 4. Similarly the limit exists. Also, $y' \to 0$ if the limit exists. Then, the same trick applies. The limit is 4 again.

(b). When $y_0 = 2$, the solution is $y(t) = \frac{4e^{4t}}{(1+t)^4 + e^{4t}}$. Hence, if $y(T) = 3.99$,
$$e^{4T} = 399(1 + T)^4$$
Here we seek the time that is on the right of $t = 0$ and it is $T \approx 2.8437$ (the $-0.91, -1.07$ are discarded because they are not the 'first time' as we increase time from $t = 0$)

(c). We compute $y(2) = \frac{4y_0e^8}{3^4(4-y_0)^3 + 3y_0e^8} \approx \frac{4y_0}{0.0272(4-y_0)^3 + 9y_0} = \frac{4y_0}{0.972890 + 0.1087}$
When this is 4.01, $y_0 \approx 4.40$. When this is 3.99, $y_0 \approx 3.66$. Hence $3.66 < y_0 < 4.40$