$y'' + py' + qy = g(t)$

1 Undetermined coefficients

This method is applied if the corresponding equation has **Constant coefficients** and the function g has a special form.For the examples, I'm just listing the forms you should try. In other words, I haven't finished the examples totally

1.1 Polynomial

If g is a polynomial P_n, we try Y=Polynomial. The degree of the polynomial we need can be determined by simply checking the degree of the left hand side. Just image you plug in a polynomial of degree m, the degree of the first term is $m-2$. The degree of the second term is $m-1$ if $p \neq 0$ while the degree is 0 if $p=0$. The degree of the third term is m if $q \neq 0$.

Hence, if $q \neq 0$, the polynomial you try should be of the same degree as P_n; otherwise, the degree should be raised.

Example 1

$$y'' + 2y' = 3t + 2$$

The polynomial we try is $Y = A_1 t^2 + A_2 t$ (no constant term since the constant solves the homogeneous equation)

Example 2

$$y'' + 2y = 3t + 2$$

The polynomial we use is $Y = A_1 t + A_2$

1.2 Exponential function multiplied by a polynomial

For exponential function multiplied by a polynomial $g = P_n \exp(rt)$, we try $Y = Q_n t^s \exp(rt)$ where Q_n is of the same degree as P_n. If we should multiply t or not depends on if $\exp(rt)$ solves the homogeneous problem or not. If it solves, we multiply t; if $t\exp(rt)$ also solves, we multiply a t further.

Example 3

$$y'' + 2y' - 3y = 2e^t$$

We see that the fundamental set is $\{e^t, e^{-3t}\}$. Here, the polynomial is of degree zero. Hence, we try $Ct^s e^t$. We see that $s = 0$ or Ce^t doesn't work because e^t solves the homogeneous problem. Hence, we use $s = 1$ or $Y = Cte^t$.

Example 4

$$y'' + 2y' - 3y = 2e^{-t}$$

We see that the fundamental set is $\{e^t, e^{-3t}\}$. Here, the polynomial is of degree zero. Hence, we try $Ct^s e^t$. $s = 0$ since e^{-t} doesn't solve the homogeneous problem. Hence, $Y = Ce^{-t}$

Example 5

$$y'' + 2y' - 3y = 2te^t$$

We see that the fundamental set is $\{e^t, e^{-3t}\}$. Here, the polynomial is of degree 1. Hence, we try $(A_1 t + A_2)t^s e^t$. We see that $s = 0$ doesn't work because e^t solves the homogeneous problem. Hence, we use $s = 1$ or $Y = (A_1 t + A_2)te^t$.

Example 6

$$y'' - 2y' + y = 2e^t$$

We see that the fundamental set is $\{e^t, te^t\}$. Here, the polynomial is of degree 0. Hence, we try $Ct^s e^t$. Since both e^t and te^t solve the homogeneous problem, we should try $s = 2$ or $Y = Ct^2 e^t$

Example 7

$$y'' - 2y' + y = t^4 e^t$$

Similar reason: $Y = (At^4 + Bt^3 + Ct^2 + D)t^2 e^t$

1.3 Polynomial times sin or cos

For $g = P_n e^{\alpha t}\cos(\beta t)$ or $g = P_n e^{\alpha t}\sin(\beta t)$, we try $Y = Q_1 t^s e^{\alpha t}\cos(\beta t) + Q_2 t^2 e^{\alpha t}\sin(\beta t)$ where Q_1 and Q_2 are of the same degree as P_n. Whether we should multiply t or not depends on whether $e^{\alpha t}\cos(\beta t)(\ldots\sin(\beta t))$ solves the homogeneous problem or not. If it solves, we multiply t; if $te^{\alpha t}\cos(\beta t)(\ldots\sin(\beta t))$ also solves, we multiply a t further.

Example 8

$$y'' + 4y = 2\sin(t)$$

The polynomial is of degree zero. Hence, we use $Y = (C_1\cos(t) + C_2\sin(t))t^s$. Notice that $\sin(t)$ doesn't solve the homogeneous equation. Then, $s = 0$, $Y = C_1\cos(t) + C_2\sin(t)$.

Example 9

$$y'' + 4y = 2\sin(2t)$$

The polynomial is of degree zero. Hence, we use $Y = (C_1\cos(2t) + C_2\sin(2t))t^s$. Notice that $\sin(2t)$ solves the homogeneous equation. Then, $s = 1$, $Y = C_1 t\cos(t) + C_2 t\sin(t)$.

Example 10

$$y'' - 2y' + 5y = te^t\cos(2t)$$

The polynomial is of degree 1. Hence, we use $Y = ((A_1 t + A_2)e^t\cos(2t) + (B_1 t + B_2)e^t\sin(2t))t^s$. Notice that $e^t\cos(2t)$ solves the homogeneous problem, hence $s = 1$.

2 Variation of parameters

Just emphasize one thing: Normalize the coefficient of y''
This method is straightforward but it may result in hard integrals.
Let's look at the suggested problems.

#2. (b). The verification is easy. We just plug in. For example, for $y_2 = t^2 \ln t$, by plugging in,

$LHS = t^2(t^2 \ln t)'' - 3t(t^2 \ln t)' + 4(t^2 \ln t) = t^2(2\ln t + 4 - 1) - 3t(2t\ln t + t) + 4t^2 \ln t = 0 = RHS$

Let's find y_p by the variation of parameters. One mistake people tend to make is to use $g = t^2 \ln t$. This is not OK. We first should normalize the coefficient of y'' to 1 The equations

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = t^2 \ln t / t^2$$

we can compute $W(y_1, y_2) = t^2(2t\ln t + t) - 2tt^2 \ln t = t^3$. We solve that

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \ln t \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ \ln t \end{bmatrix}$$

Hence, $u_1' = \frac{-y_2 \ln t}{W} = -(\ln t)^2/t$, which yields $u_1 = -\int \frac{1}{t}(\ln t)^2 dt = -\frac{1}{3}(\ln t)^3$(we only need one and hence the constant is ignored).

Similarly, $u_2' = \frac{1}{t}\ln t$ and thus $u_2 = \frac{1}{2}(\ln t)^2$.

$$y_p = t^2(-\frac{1}{3}(\ln t)^3) + t^2 \ln t(\frac{1}{2}(\ln t)^2) = \frac{1}{6}t^2(\ln t)^3$$

(a). $t^2 y'' - 2y = 3t^2 - 1$. $\{t^2, t^{-1}\}$ (Note: the homogeneous problem can be solved by plugging in t^n since it's an Euler's ODE)

$Y = u_1 t^2 + u_2 t^{-1}$.

In our case, $g = (3t^2 - 1)/t^2$.

$$W = t^2(t^{-1})' - 2t * t^{-1} = -3$$

Hence,

$$u_1' = -\frac{y_2 g}{W} = \frac{1}{3}(3t^{-1} - t^{-3}) \Rightarrow u_1 = \ln t + \frac{1}{6}t^{-2}$$

and

$$u_2' = \frac{y_1 g}{W} = -\frac{1}{3}(3t^2 - 1) \Rightarrow u_2 = -\frac{1}{3}t^3 + \frac{1}{3}t$$

Hence,

$$Y = u_1 y_1 + u_2 y_2 = t^2 \ln t - \frac{1}{3}t^2 + \frac{1}{2}$$

Notice that $-t^2/3$ is a solution of the homogeneous equation which can be thrown away safely. Hence, we can simply pick

$$Y = t^2 \ln t + \frac{1}{2}$$

but the previous expression is also correct.