Keys-Quiz 8

1. Consider $f(x,y) = \ln(1+x)2^y$.

   (a). Compute the Taylor expansion of f about $(0,0)$ up to second order.

   (b). Let $g(x,y) = \ln(1+x^2)2^{y^3}$. Can you figure out $g_{xx}(0,0), g_{xy}(0,0), g_{yy}(0,0)$ without computing $g_{xx}(x,y), g_{xy}(x,y), g_{yy}(x,y)$? (Hint: No computation is needed. Use the result in (a) and do substitution.)

   Soln. (a). We first compute the following formulas:

   $$f_x = \frac{1}{x+1}2^y \qquad f_y = \ln(1+x)2^y \ln 2$$

   $$f_{xx} = -\frac{1}{(x+1)^2}2^y \quad f_{xy} = \frac{1}{x+1}2^y \ln 2 \quad f_{yy} = \ln(x+1)2^y(\ln 2)^2$$

   The Taylor expansion at $(0,0)$ is $f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2}f_{xx}(0,0)(x-0)^2 + f_{xy}(0,0)(x-0)(y-0) + \frac{1}{2}f_{yy}(0,0)(y-0)^2$.

   $$f(x,y) \approx 0 + 1(x-0) + 0(y-0) + \frac{1}{2}(-1)(x-0)^2 + \ln 2(x-0)(y-0) + \frac{1}{2}0(y-0)^2$$

   $$f(x,y) \approx x - \frac{1}{2}x^2 + (\ln 2)xy$$

   (b). By the formula in (a), we have

   $$g(x,y) = f(x^2, y^3) \approx x^2 - \frac{1}{2}x^4 + (\ln 2)x^2 y^3 + \ldots$$

   We only need to keep the terms up to second order and hence

   $$g(x,y) \approx x^2$$

   From here, we read that $\frac{1}{2}g_{xx}(0,0) = 1, g_{xy}(0,0) = 0, \frac{1}{2}g_{yy}(0,0) = 0$

2. $f(x,y)$ is differentiable(and hence continuous) and satisfies that $f_x(x,y) = f_y(x,y) > 0$ for all points (*Caution: this doesn't mean $f_x = f_y$ is a constant*). Inside the unit disk $x^2 + y^2 \leq 1$, does f have a global maximum and a global minimum? If yes, find them. (This is an old exam problem. For the boundary extremum, use Lagrange multiplier.)

   Soln. The domain is the unit disk which is bounded and closed, and thus f, which is continuous, must have a global max and a global minimum on this disk.

   If any of them appears in the interior($x^2 + y^2 < 1$), we must have $\nabla f = \vec{0}$ there. However, as $f_x = f_y > 0$, this is impossible. Hence, the global max and global min must appear on the boundary $x^2 + y^2 = 1$.

   We are solving max / min f with constraint $g(x,y) = x^2 + y^2 = 1$ now. Notice $\nabla g \neq \vec{0}$ on $x^2 + y^2 = 1$ and thus we have

   $$\nabla f = \lambda \nabla g = \lambda(2x, 2y) \quad x^2 + y^2 = 1$$

Since $f_x = f_y$, we must have $2\lambda x = 2\lambda y$. You can see that $\lambda \neq 0$, for otherwise $f_x = 0$. Hence $x = y$. You can then get two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. One of them must be maximum and one of them must be minimum. If you recall $f_x > 0, f_y > 0$, you can figure out that $(-1/\sqrt{2}, -1/\sqrt{2})$ is the minimum point while $(1/\sqrt{2}, 1/\sqrt{2})$ is the maximum point.

(Bonus 2. 1 pt for each) Solve the integrals (I'm not expecting you to solve them all. Choose the ones you feel good to solve.)

a. $\int \frac{\sqrt{y^2 - 49}}{y} dy$

b. $\int_0^1 \ln x dx$

c. $\int \frac{1}{(x+1)(x^2+1)} dx$

Soln. a. In the square root, you have $y^2 - 49 = y^2 - a^2$. Recalling the identity $\sec^2 \theta - 1 = \tan^2 \theta$, you can use $y = 7 \sec \theta$. Then, $dy = 7 \sec \theta \tan \theta d\theta$.

$$\int \frac{7\sqrt{\sec^2 \theta - 1}}{7 \sec \theta} 7 \sec \theta \tan \theta d\theta = \int 7 \tan^2 \theta d\theta = 7 \int (\sec^2 \theta - 1) d\theta = 7 \tan \theta - 7\theta + C$$

Now, we should write them in terms of y. If $7 \sec \theta = y$, then you can make the hypotenuse y and the adjacent edge to be 7. The opposite edge is $\sqrt{y^2 - 49}$. Hence $\tan \theta = \sqrt{y^2 - 49}/7$. For θ, you have $\sec \theta = y/7$ and thus $\cos \theta = 7/y$ or $\theta = \arccos(7/y)$. The answer is

$$\sqrt{y^2 - 49} - 7 \arccos(\frac{7}{y}) + C$$

b. We do integration by parts: $\int u dv = uv - \int v du$. In our case, $u = \ln x, v = x$. Hence you have

$$\int \ln x dx = (\ln x)x - \int x d(\ln x) = x \ln x - \int 1 dx = x \ln x - x + C$$

The improper integral is

$$\int_0^1 \ln x dx = \lim_{b \to 0} (1 \ln 1 - 1 - b \ln b + b) = -1$$

Informally, you may write $x \ln x|_0^1 - \int_0^1 x \frac{1}{x} dx = 0 - \lim_{x \to 0} x \ln x - \int_0^1 dx$

c. Notice $x^2 + 1$ is an irreducible quadratic factor. You then have the partial fraction

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Then, $A = C = 1/2, B = -1/2$.

$$\frac{1}{2} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx = \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan(x) + C$$