Keys-Quiz 7

1. $f(x, y) = x^3 + y^3 + 3xy$. Find all critical points and classify them into local maxima, local minima and saddle points.

   Soln. For critical points, we have

   $$f_x = 3x^2 + 3y = 0$$
   $$f_y = 3y^2 + 3x = 0$$

   From the first equation, we have $y = -x^2$. Plug this into the second, we get $3(-x^2)^2 + 3x = 0$ or $x(x^3 + 1) = 0$. Hence, we have $x = 0$ or $x = -1$. For $x = 0$, we get $y = -0^2 = 0$; for $x = -1$, we get $y = -(-1)^2 = -1$. We have two critical points $(0, 0)$ and $(-1, -1)$.

   To classify them, we use 2nd derivative test. We compute $f_{xx} = 6x; f_{xy} = 3; f_{yy} = 6y$. The quadratic form is $\frac{1}{2} f_{xx} \Delta x^2 + f_{xy} \Delta x \Delta y + \frac{1}{2} f_{yy} \Delta y^2$ and we only need to check the number $f_{xx} f_{yy} - f_{xy}^2$.

   At point $(0, 0)$, this number is $0*0-3^2 < 0$ and the form is indefinite. $(0, 0)$ is a saddle point. At $(-1, -1)$, this number is $(-6) * (-6) - 3^2 = 27 > 0$. We also notice $f_{xx}/2 = -3 < 0$. The form here is negative definite. Then $(-1, -1)$ is a local max.

2. $f(x, y, z) = x + 2y + 4z$ has a smallest value on the surface $xyz = 1, x > 0, y > 0, z > 0$. Find the value. (Hint: Lagrange multiplier. Argue x, y, z are nonzero so that $1/x$ is safe. Discuss $\nabla g = 0$ also(so g?))

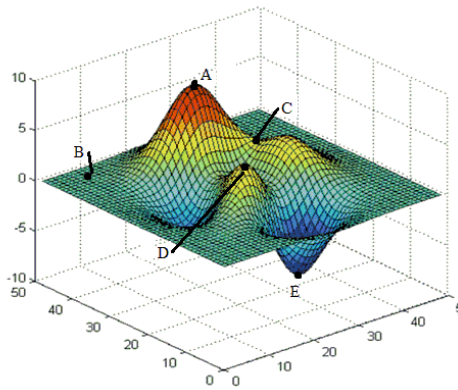
   Soln. Let $g(x, y, z) = xyz = 1$. We consider two cases $\nabla g = 0, g = 1$ and $\nabla f = \lambda \nabla g, g = 1$. Notice that $\nabla g = (yz, xz, xy)$. If $\nabla g = 0$, then one of x, y, z must be zero, which contradicts with $xyz = 1$. That means we only have the second case.

   Writing out the second case, we have

   $$1 = \lambda yz$$
   $$2 = \lambda xz$$
   $$4 = \lambda xy$$
   $$xyz = 1$$

   Notice that $xyz = 1$ implies $yz = 1/x$ since $x \neq 0$. The first equation is $1 = \lambda/x$ or $x = \lambda$. Similarly, the second equation tells you $2y = \lambda$ and the third equation tells you $4z = \lambda$. In other words, we have $x = 2y = 4z$. Using $xyz = 1$, we solve $x = 2, y = 1, z = 1/2$. We only have one candidate. We know the minimum point exists and this must be the minimum point. The values is $2 + 2(1) + 4(1/2) = 6$

(Bonus 1: 2pts) In the figure, $B(8, 42, 0)$ and for all $(x, y), x \leq 8, y \geq 42$, the function value is 0(namely the function is all zero on the left-upper corner of B). Both B and E are interior local minimum points. Consider the quadratic form $Q(\Delta x, \Delta y) = \frac{1}{2} f_{xx}(a, b)\Delta x^2 + f_{xy}(a, b)\Delta x \Delta y + \frac{1}{2} f_{yy}(a, b)\Delta y^2$. Which kind of form could Q be at B? How about Q at E? Explain.



Soln. Since the points are minimum points, the form can't be negative definite(otherwise, local max), indefinite(otherwise, saddle). We are left with two cases: semidefinite and positive definite. It's clear that at E the graph is like a bowl and the form there should be positive definite. At B, the graph is kind of flat, which is not like a bowl. The only possibility is that it's semidefinite at B. (Actually, you can determine $f_{xx} = f_{xy} = f_{yy} = 0$)

(Bonus 2: 2pts) (a) Explain briefly why $\nabla f \parallel \nabla g$ at the point where f achieves one extremum on $g = C$. (b). Consider the second regular problem. I'm wondering how to find the maximum value. By solving $\nabla f = \lambda \nabla g, g = C$ and $\nabla g = 0, g = C$, I get one point only. We have seen that it's the minimum value point. I'm wondering where the maximum point goes. Please help me.

Soln. (a). Ommited. See your notes for lecture and discussion. The simplest way is to say the two level sets are tangent to each other at the extremum. (b). The reason is that there's no maximum point. For example, $x = 0.0001, y = 0.0001$, your $z = 1/(xy) = 10^8$ and $f = x + 2y + 4z$ is large. This can keep going. Now you see the importance of the theorem in this chapter: *if the domain is bounded and closed, then we are guaranteed that there's a maximum and there's a minimum.* Here, $g = C$ is not a bounded domain. If you know there's an extremum, then it must be from the points you find by solving Lagrange multiplier equations.