Keys-Quiz 5

1. (6) $F(x, y, z) = e^{x-y} + \ln z - z^2$. Consider the **zero set** of this function that passes through $(x_0, 1, 1)$.

   (a). Determine x_0 value and compute ∇F at this point

   (b). Compute the tangent plane of the zero set at point $(x_0, 1, 1)$.

   Soln. (a). The zero set condition implies that

$$e^{x_0-1} + \ln 1 - 1^2 = 0 \Rightarrow e^{x_0-1} = 1 \Rightarrow x_0 = 1$$

The gradient is

$$\nabla F(x, y, z) = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} e^{x-y} \\ -e^{x-y} \\ \frac{1}{z} - 2z \end{pmatrix}$$

Evaluating it at $(1, 1, 1)$, we get

$$\nabla F(1, 1, 1) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

(b). $\nabla F(1, 1, 1)$ is a normal vector of the tangent plane and thus the tangent plane is

$$\nabla F(1, 1, 1) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ 1 \\ 1 \end{pmatrix} \right) = 0$$

or $1(x - 1) - 1(y - 1) - 1(z - 1) = 0$

Comment: Some students made the following mistakes:

A. tangent $= \nabla F \cdot ((x, y, z) - (1, 1, 1))$. *This is not true because you should give an equation for the tangent plane. Written like this, it's not even an equation. You should have* $\nabla F \cdot ((x, y, z) - (1, 1, 1)) = 0$

B. Some people had

$$\begin{pmatrix} e^{x-y} \\ -e^{x-y} \\ \frac{1}{z} - 2z \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = 0$$

and got complicated $e^{x-y}(x - 1) + \ldots = 0$ *etc. This is also wrong since we need the normal vector* ∇F *of the tangent plane at* $(1, 1, 1)$ *only. You should evaluate the gradient at* $(1, 1, 1)$*. The* x *value in the gradient is just* 1*. It's different from the* x *in the expression* $(x - 1)$*. The* $'x'$ *in* $'(x - 1)'$ *is the* x *coordinate of an arbitrary point on the plane, which can be any value, not just* 1*.*

2. (4) $f(x,y) = \ln(2+2x+e^y)$. Let's say C is the level set of f passing through $(1,0)$. Locally around $(1,0)$, could we regard the level set C as the graph of an implicit function $y = g(x)$? If yes, compute $dy/dx|_{x=1} = g'(1)$.

Comment: I have rewritten the solution in a more organized way. In exam, you can follow this way to organize you solution.

Soln. We regard y as a function of x: $y = y(x)$. Thus, we have

$$f(x, y(x)) = c$$

Taking derivative on x, we have:

$$f_x(x, y(x)) + f_y(x, y(x)) * \frac{dy}{dx} = 0$$

Let's check

$$f_y = \frac{e^y}{2 + 2x + e^y}$$

Evaluating this at $x = 1, y(1) = 0$ (this is because the level set passes through $(1, 0)$ and the y coordinate should be the function value if the function exists), we see

$$f_y(1,0) = \frac{1}{2+2+1} = \frac{1}{5} \neq 0$$

The function exists locally.

Solving the above yields

$$g'(x) = \frac{dy}{dx} = -\frac{f_x(x, y(x))}{f_y(x, y(x))}$$

Let's compute f_x:

$$f_x = \frac{2}{2 + 2x + e^y} \Rightarrow f_x(1,0) = \frac{2}{5}$$

The derivative is therefore:

$$g'(1) = \frac{dy}{dx}|_{x=1} = -\frac{2/5}{1/5} = -2$$

(Bonus: 2 pts) Consider again $f(x, y) = \ln(2 + 2x + e^y)$ and the level set C that passes through $(1, 0)$. Compute the tangent line of C at $(1, 0)$ in two ways:

- Using $y = g(1) + g'(1) * (x - 1)$
- Using the fact that ∇f is perpendicular with the tangent line and $\nabla f \cdot (\vec{x} - \vec{x}_0) = 0$

Verify that they agree.

Comment: This is true for $z = f(x, y)$ as well. The tangent plane computed using $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ should agree with the tangent plane for the level set $F(x, y, z) = f(x, y) - z = 0$

Soln.

From the solution above, we see that $g(1) = 0$ and $g'(1) = -2$. The tangent line is

$$y = 0 - 2(x - 1)$$

Using the other way,

$$\nabla f = \begin{pmatrix} 2/(2 + 2x + e^y) \\ e^y/(2 + 2x + e^y) \end{pmatrix}$$

Evaluating at $(1, 0)$, we have

$$\nabla f = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix}$$

The tangent line is

$$\begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0$$

This gives you $2(x - 1) + y = 0$ which agrees with the above.