Key-Quiz 3(Version 2)

Consider the parabola $y = x^2$.

1. (6) One can use $x = t$ to parametrize the curve and get $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

   The curvature κ could be computed using $\frac{1}{\|\vec{x}'(t)\|^3}\|\vec{x}'(t) \times \vec{x}''(t)\|$. Find this curvature expression. (Notice that this formula only gives you the curvature but not the normal.)

   Soln. If we pick $x(t) = t$, then $y(t) = x(t)^2 = t^2$. Therefore, the parametrization is

   $$\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

   It's easy to compute $\vec{x}'(t) = (1, 2t)^T$ and $\vec{x}''(t) = (0, 2)^T$ where T means transpose. Then, we have $\|\vec{x}'(t)\| = \sqrt{1+4t^2}$ and

   $$\vec{x}'(t) \times \vec{x}''(t) = \hat{k}\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} = 2\hat{k}$$

   where the cross product is computed by adding 0 in the third column.

   By the formula, the curvature is therefore

   $$\kappa = \frac{\|2\hat{k}\|}{\sqrt{1+4t^2}^3} = \frac{2}{(1+4t^2)^{3/2}}$$

2. (4) Set up the integral of the arclength for the portion between $(0,0)$ and $(1,1)$. You don't have to solve the integral but explain to me which technique of integration is suitable.

   Soln. At $(0,0)$, $t = 0$ and at $(1,1)$, $t = 1$. Then, the arclength is

   $$L = \int_0^1 \|\vec{x}'(t)\|dt = \int_0^1 \sqrt{1+4t^2}dt$$

   This integral could be finished by trig substitution $2t = \tan\theta$

3. (Bonus:2) Suppose θ is the angle between the tangent and x-axis. The change of θ is given by $d\theta = \kappa ds = \kappa\|\vec{x}'(t)\|dt$. Compute the total change of the angle of the parabola $\int_{-\infty}^{\infty} \kappa\|\vec{x}'(t)\|dt$ and explain your answer.

   Soln. (An interesting fact is that this θ is exactly the one in the trig sub above... Anyway, let's compute...)

   By the curvature we have already. the total change is

   $$\Delta\theta = \int_{-\infty}^{\infty} \frac{2}{(1+4t^2)^{3/2}}\sqrt{1+4t^2}dt = \int_{-\infty}^{\infty} \frac{2}{1+4t^2}dt = \arctan(2t)|_{-\infty}^{\infty} = \pi$$

   At $-\infty$, the tangent is pointing downward and $\theta(-\infty) = -\pi/2$ and at ∞, the tangent is pointing upward and $\theta(\infty) = \pi/2$. The difference is exactly π.