234 Quiz 3-Keys

1. Let $\vec{x}(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}$. Compute the curvature κ and the unit normal vector $\vec{N}$. (Hint: you may want to use $\|\lambda\vec{a}\| = |\lambda|\|\vec{a}\|$)

   Soln. Recall the formula $\vec{\kappa} = \frac{d}{ds}\vec{T} = \frac{1}{\|\vec{x}'(t)\|}\frac{d}{dt}\vec{T}$. We need the unit tangent first:

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \frac{1}{\sqrt{1+4t^2}}\begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

   The the curvature vector is

$$\begin{aligned}
\vec{\kappa}(t) &= \frac{1}{\|\vec{x}'(t)\|}\frac{d}{dt}\vec{T} = \frac{1}{\sqrt{1+4t^2}}\left(\frac{1}{\sqrt{1+4t^2}}\begin{pmatrix} 2t \\ 1 \end{pmatrix}\right)' \\
&= \frac{1}{\sqrt{1+4t^2}}\left(-4t\frac{1}{(1+4t^2)^{3/2}}\begin{pmatrix} 2t \\ 1 \end{pmatrix} + \frac{1}{(1+4t^2)^{1/2}}\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) \\
&= \frac{1}{(1+4t^2)}\left(\frac{1}{1+4t^2}\begin{pmatrix} -8t^2 \\ -4t \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \frac{1}{(1+4t^2)^2}\begin{pmatrix} 2 \\ -4t \end{pmatrix}
\end{aligned}$$

   The curvature is then the magnitude of the curvature vector and thus

$$\kappa = \|\vec{\kappa}\| = \frac{2}{(1+4t^2)^2}\|\begin{pmatrix} 1 \\ -2t \end{pmatrix}\| = \frac{2}{(1+4t^2)^{3/2}}$$

   The normal is just the direction of the curvature vector which is also the direction of $\begin{pmatrix} 1 \\ -2t \end{pmatrix}$ and hence

$$\vec{N} = \frac{1}{\|\begin{pmatrix} 1 \\ -2t \end{pmatrix}\|}\begin{pmatrix} 1 \\ -2t \end{pmatrix} = \frac{1}{\sqrt{1+4t^2}}\begin{pmatrix} 1 \\ -2t \end{pmatrix}$$

   The next problem is on next page

2. (6) Let $\vec{x}(t) = \begin{pmatrix} t^2 \cos t \\ t^2 \sin t \\ t^2 \end{pmatrix}$ where $t \geq 0$. Compute the acceleration at $t = 0$ and the arclength of the portion from $t = 0$ to $t = 1$.

Soln. We compute the velocity as

$$\vec{v}(t) = \vec{x}'(t) = \begin{pmatrix} 2t\cos t - t^2 \sin t \\ 2t \sin t + t^2 \cos t \\ 2t \end{pmatrix}$$

The acceleration is $\vec{a}(t) = \vec{v}'(t)$. Notice that at $t = 0$, all the terms containing t would vanish. Therefore, the derivative of the first entry at $t = 0$ only leaves us $2\cos 0 + 0(\dots) = 2$. Similarly, the derivative of the second entry yields $2 \sin 0 + 0(\dots) = 0$. The third entry is 2. Thus the acceleration at $t = 0$ is

$$\vec{a}(0) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

The speed is

$$\|\vec{x}'(t)\| = \sqrt{(2t\cos t - t^2 \sin t)^2 + (2t \sin t + t^2 \cos t)^2 + 4t^2}$$

$$= \sqrt{(4t^2 \cos^2 t - 4t^3 \cos t \sin t + t^4 \sin^2 t) + (4t^2 \sin^2 t + 4t^3 \cos t \sin t + t^4 \cos^2 t) + 4t^2}$$

$$= \sqrt{4t^2 + t^4 + 4t^2} = \sqrt{8t^2 + t^4} = t\sqrt{t^2 + 8}$$

If you noticed, you can pull out t at the very beginning. $\vec{v} = t(\dots)$ *and compute the magnitude of the remaining part.* The arc length is therefore

$$L = \int_0^1 t\sqrt{t^2 + 8}dt = \frac{1}{2}\int_{u=8}^{u=9} \sqrt{u}du$$

where $u = t^2 + 8$. The the integral is $\frac{1}{3}u^{3/2}|_8^9 = \frac{1}{3}(3^3 - 8^{3/2}) = 9 - \frac{16}{3}\sqrt{2}$