1. Let \( \mathbf{x}(t) = \left( t^2, t \right) \). Compute the curvature \( \kappa \) and the unit normal vector \( \mathbf{N} \). (Hint: you may want to use \( \| \lambda \mathbf{a} \| = |\lambda| \| \mathbf{a} \| \))

Soln. Recall the formula \( \kappa = \frac{d}{ds} \mathbf{T} = \frac{1}{\| \mathbf{x}'(t) \|} \frac{d}{dt} \mathbf{T} \). We need the unit tangent first:

\[
\mathbf{T}(t) = \frac{\mathbf{x}''(t)}{\| \mathbf{x}''(t) \|} = \frac{1}{\sqrt{1 + 4t^2}} \left( \begin{array}{c} 2t \\ 1 \end{array} \right)
\]

The the curvature vector is

\[
\kappa(t) = \frac{1}{\| \mathbf{x}''(t) \|} \frac{d}{dt} \mathbf{T} = \frac{1}{\sqrt{1 + 4t^2}} \left( \frac{1}{\sqrt{1 + 4t^2}} \left( \begin{array}{c} 2t \\ 1 \end{array} \right) \right)'
\]

\[
= \frac{1}{\sqrt{1 + 4t^2}} \left( -4t \frac{1}{(1 + 4t^2)^{3/2}} \left( \begin{array}{c} 2t \\ 1 \end{array} \right) + \frac{1}{(1 + 4t^2)^{1/2}} \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \right)
\]

\[
= \frac{1}{(1 + 4t^2)} \left( \frac{1}{1 + 4t^2} \left( \begin{array}{c} -8t^2 \\ -4t \end{array} \right) + \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \right) = \frac{1}{(1 + 4t^2)} \left( \begin{array}{c} 2 \\ -4t \end{array} \right)
\]

The curvature is then the magnitude of the curvature vector and thus

\[
\kappa = \| \kappa(t) \| = \frac{2}{(1 + 4t^2)^{3/2}} \left( \begin{array}{c} 1 \\ -2t \end{array} \right) || = \frac{2}{(1 + 4t^2)^{3/2}}
\]

The normal is just the direction of the curvature vector which is also the direction of \( \left( \frac{1}{-2t} \right) \) and hence

\[
\mathbf{N} = \frac{1}{\| \left( \frac{1}{-2t} \right) \|} \left( \begin{array}{c} 1 \\ -2t \end{array} \right) = \frac{1}{\sqrt{1 + 4t^2}} \left( \frac{1}{-2t} \right)
\]

The next problem is on next page
2. (6) Let \( \vec{x}(t) = \begin{pmatrix} t^2 \cos t \\ t^2 \sin t \\ t^2 \end{pmatrix} \) where \( t \geq 0 \). Compute the acceleration at \( t = 0 \) and the arclength of the portion from \( t = 0 \) to \( t = 1 \).

Soln. We compute the velocity as
\[
\vec{v}(t) = \vec{x}'(t) = \begin{pmatrix} 2t \cos t - t^2 \sin t \\ 2t \sin t + t^2 \cos t \\ 2t \end{pmatrix}
\]
The acceleration is \( \vec{a}(t) = \vec{v}'(t) \). Notice that at \( t = 0 \), all the terms containing \( t \) would vanish. Therefore, the derivative of the first entry at \( t = 0 \) only leaves us \( 2 \cos 0 + \ldots = 2 \). Similarly, the derivative of the second entry yields \( 2 \sin 0 + \ldots = 0 \). The third entry is 2. Thus the acceleration at \( t = 0 \) is
\[
\vec{a}(0) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}
\]
The speed is
\[
\| \vec{x}'(t) \| = \sqrt{(2t \cos t - t^2 \sin t)^2 + (2t \sin t + t^2 \cos t)^2 + 4t^2}
\]
\[
= \sqrt{(4t^2 \cos^2 t - 4t^3 \cos t \sin t + t^4 \sin^2 t) + (4t^2 \sin^2 t + 4t^3 \cos t \sin t + t^4 \cos^2 t) + 4t^2}
\]
\[
= \sqrt{4t^2 + t^4 + 4t^2} = \sqrt{8t^2 + t^4} = t \sqrt{t^2 + 8}
\]
If you noticed, you can pull out \( t \) at the very beginning. \( \vec{v} = t(\ldots) \) and compute the magnitude of the remaining part. The arc length is therefore
\[
L = \int_0^1 t \sqrt{t^2 + 8} dt = \frac{1}{2} \int_{u=8}^{u=9} \sqrt{u} du
\]
where \( u = t^2 + 8 \). The the integral is \( \frac{1}{2} u^{3/2}\big|_8^9 = \frac{1}{2} \left(3^3 - 8^{3/2}\right) = 9 - \frac{16}{3} \sqrt{2} \)