Keys-Quiz 11

1. (6) Let $\mathcal{C}$ be the boundary of the region bounded by $y = 0, x = 3, y = x$ and oriented counterclockwisely. Consider $\vec{v} = (y^2 - x^2, x^2 + y^2)$.

   (a). Compute the circulation $\oint_{\mathcal{C}} \vec{v} \cdot d\vec{x}$

   We apply Green's Theorem:

   $$\oint_{\mathcal{C}} \vec{v} \cdot d\vec{x} = \oint_{\mathcal{C}} (y^2 - x^2)dx + (x^2 + y^2)dy = \iint_D (-2y + 2x)dA$$

   The region is well-suited for Cartesian coordinate. $0 \leq x \leq 3, 0 \leq y \leq x$ and hence

   $$\int_0^3 \int_0^x (-2y + 2x)dydx = \int_0^3 x^2 dx = 9$$

   (b). Write out the formula for the outer flux of the field along the curve and change it into double integral without solving.

   The formula is $\oint_{\mathcal{C}} \vec{v} \cdot \vec{N} ds$. If you like, you can plug in $\vec{N} ds = (dy, -dx)$

   Green's Theorem tells us the outer flux equals

   $$\iint_D ((y^2 - x^2)_x + (x^2 + y^2)_y)dA = \iint_D (-2x + 2y)dxdy = \int_0^3 \int_0^x (-2x + 2y)dydx$$

2. (4) Consider $\vec{F} = (e^y, xe^y)$ and $\mathcal{C}$: $y = \sqrt{x^5 + 1}$, $x : 0 \to 2$. Find $\int_{\mathcal{C}} \vec{F} \cdot d\vec{x}$

   You can parametrize $\vec{x}(t) = (t, \sqrt{t^5 + 1}), 0 \leq t \leq 2$ but the computation is not so easy. Instead, we notice that the filed is conservative as $(e^y)_y = (xe^y)_x$ and thus $\vec{F} = \nabla f$. *Notice that the curve is NOT closed, so even if the field is conservative, the integral may not be zero.* We find f by integrating

   $$f = \int e^y dx = xe^y + g(y)$$

   Using the fact that $f_y = xe^y$, we determine $g(y) = C$. Hence, the integral is simply

   $$\int_{\mathcal{C}} \nabla(xe^y) \cdot d\vec{x} = xe^y|_{(0,1)}^{(2,\sqrt{33})} = 2e^{\sqrt{33}}$$

   Bonus on back.

(Bonus) $\vec{v} = (P, Q) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$ and $\mathcal{C}$ is the unit circle oriented counterclockwisely.

(a).(1.5) The circulation $\oint_{\mathcal{C}} \vec{v} \cdot d\vec{x} = 2\pi$. However, one is tempted to use Green's Theorem and get $\iint_D (-P_y + Q_x) dA$. $-P_y + Q_x = 0$. Does this example say the Green's Theorem is wrong, why?

The Green's Theorem is not wrong. The point is that we can't apply Green's theorem directly here since $\vec{v}$ is not well-defined at the origin which is inside the curve.

If one still wants to apply Green's theorem, there are two ways out. The first way is to isolate the singular point with a small circle. Then apply Green's theorem to the region that doesn't contain the interior of the small disk. Then, you can find the relation between the integral on the outer circle and the integral on the inner circle. Finally, one takes the limit that the radius of the inner circle goes to zero; Another way in physics is to notice that the curl is actually a Dirac-delta function: $-P_y + Q_x = 2\pi\delta(\vec{x})$.

(b).(1.5) You see that the circulation is nonzero, so it's not conservative in the whole domain where it's defined. Actually $\vec{v} = \nabla\theta$. Does this contradict with the fact that the gradient is always conservative?

It's not a contradiction. If you look at the field, the domain where it's defined is not simply connected. If you look at the function, θ is not a good function. A function needs to be single valued. To achieve the single-value requirement, you must cut the plane. However, as you cut the plane, θ has jump at the cut and it's not continuous at the cut. For example, you may cut on positive x-axis. Then, there's jump 2π at the cut. Apply the fundamental theorem: $\oint_{\mathcal{C}} \nabla\theta \cdot d\vec{x} = \theta(1, 0^-) - \theta(1, 0^+) = 2\pi$. Having the cut, the domain $\mathbb{R}^2 \setminus \{(x, 0), x \geq 0\}$ is simply connected and the function θ is good. The field becomes conservative under this modified domain.