234 Quiz 10

Section: *Name:*

20 minutes. You can get 10 points at most.

1. Consider the parabola $y = x^2$. Let $\mathcal{C}$ be the part from $y = 1$ to $y = 4$ in the first quadrant.

   (a). (5) Compute $\int_{\mathcal{C}} x ds$.

   Parametrize the curve as $\vec{x}(t) = (t, t^2), 1 \le t \le 2$ because when $y = 4$, $t^2 = 4, t = 2$.

   Then, we see that $d\vec{x} = (1, 2t)dt$ and $ds = |\vec{x}'(t)|dt = \sqrt{1+4t^2}dt$

   Using $u = 4t^2 + 1$,

$$\int_1^2 t\sqrt{1+4t^2}dt = \int_5^{17} \frac{1}{8}\sqrt{u}du = \frac{1}{12}u^{3/2}|_5^{17}$$

   (b). (1) Is $\int_{\mathcal{C}} xydx = \frac{1}{2}x^2y|_{(1,1)}^{(2,4)} = \frac{1}{2}4*4 - \frac{1}{2}1*1$ correct or wrong?

   It's wrong. y is not a constant along the curve.

2. (5) Consider the gravitational potential energy of a satellite near the Earth $\phi = -D/r$ where r is the distance from the center of the Earth. (By physics, $D = GMm$ is a constant but you don't need this expression here.) The gravitational force is $\vec{F} = -\nabla\phi$. Let $\mathcal{C}$ be the line segment from $(1, 1, 1)$ to $(2, 2, 2)$. Compute $\int_{\mathcal{C}} \vec{F} \cdot d\vec{x}$ and verify it's $\phi(1, 1, 1) - \phi(2, 2, 2)$.

   We first parametrize the curve and compute $d\vec{x}$, which has 2 pts. The line segment can be computed as $\vec{x}(t) = (1, 1, 1) + (2-1, 2-1, 2-1)t = (1+t, 1+t, 1+t), 0 \le t \le 1$. Then, $d\vec{x} = (1, 1, 1)dt$.

   Now, let's compute the gradient, which has 2 pts. You should know $r = \sqrt{x^2+y^2+z^2}$ and the chain rule.$\nabla\phi = D(x, y, z)/r^3$ because

$$\phi_x = \phi_r r_x = \frac{D}{r^2}r_x = \frac{D}{r^2}\frac{x}{r} = \frac{Dx}{r^3}$$

   Let's check: $r_x = [\sqrt{x^2+y^2+z^2}]_x = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2+z^2}}2x = \frac{x}{r}$

   With the parametrization, $r = \sqrt{(1+t)^2 + (1+t)^2 + (1+t)^2} = \sqrt{3}(t+1)$. Hence the integral is

$$-\int_0^1 \frac{D}{3\sqrt{3}(1+t)^3}\begin{pmatrix} 1+t \\ 1+t \\ 1+t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt = -\int_0^1 \frac{Ddt}{\sqrt{3}(1+t)^2} = \frac{D}{\sqrt{3}(1+t)}|_0^1 = \phi(1,1,1) - \phi(2,2,2)$$