

## 1 Basic concepts

1. Understand the fundamental theorem:  $\nabla f \cdot d\vec{x} = df$ . Then  $\int_C \nabla f \cdot d\vec{x} = \int_C df$  is just the total change of  $f$  values.
2. Conservative field:  $\vec{F} = (P, Q)$ :  $\oint \vec{F} \cdot d\vec{x} = 0$  for any closed curve  $C$ . If it's conservative, then it's the gradient of something. Clairaut's theorem says that  $P_y = Q_x$ . Use this to check if it's conservative or not. Consequences: (1). Evaluating  $\int_C \vec{F} \cdot d\vec{x}$ , where  $C$  is not closed, is easy if you use the fundamental theorem. (2). The value of line integral only depends on the endpoints, not depending on how you arrive there.

*Comment: In 3D case, we use  $\nabla \times \vec{F} = \vec{0}$  to check if it's conservative or not.*

3. The outer normal (Draw the picture of a curve and the outer normal.) The flux integral (line integral version)  $\int_C \vec{v} \cdot \vec{N} ds$  describes the amount of material flowing across the curve in unit time.

In the 2D case, we know  $d\vec{x} = (dx, dy) = \vec{T} ds$ . Then,  $\vec{N} ds = \vec{T} \times \hat{z} ds = d\vec{x} \times \hat{z} = (dx, dy) \times \hat{z} = (dy, -dx) = (y'(t), -x'(t)) dt$ . (For 3D, flux integral is defined using surface integral.)

## 2 Exercises

1. Let  $C$  be  $y = \sin x$  from  $(0, 0)$  to  $(\frac{\pi}{2}, y_1)$ .
  - (a). Set up the integral without solving:  $\int_C xy dy$
  - (b). Let  $\vec{F} = (y, x)$ . Compute  $\int_C \vec{F} \cdot d\vec{x}$
2. Let  $C_1$  be the dogleg path from  $(0, 0)$  to  $(1, 1)$  through  $(0, 1)$ , and  $C_2$  be the dogleg path from  $(0, 0)$  to  $(1, 1)$  through  $(1, 0)$ .
  - (a). Is  $\vec{F} = (2xy, x^2)$  a conservative field? Evaluate  $\int_{C_1} \vec{F} \cdot d\vec{x}$ . Is this equal to  $\int_{C_2} \vec{F} \cdot d\vec{x}$ ? (If it's conservative, use the fundamental theorem.)
  - (b). Let  $\vec{F} = (x^2 y, 0)$ . Answer the same questions.
3. (a). Let  $C$  be the line segment from  $(1, 1, 1)$  to  $(2, 2, 2)$ . Parametrize it and compute  $d\vec{x}$  using your parametrization.
  - (b). Consider  $\phi(x, y, z) = -D/r$  where  $D$  is a constant and  $r = \sqrt{x^2 + y^2 + z^2}$  is the distance from the origin. Compute  $\vec{F} = -\nabla \phi$  using the chain rule  $\phi_x = \phi_r r_x$  etc and the fact  $r_x = x/\sqrt{x^2 + y^2 + z^2} = x/r$
  - (c). Is  $\vec{F}$  a conservative field in the first octant? Use the parametrization in (a) and the fundamental theorem to compute  $\int_C \vec{F} \cdot d\vec{x}$ . Verify that the two methods agree.
4. Exercises about flux integrals? Wait until I come back...