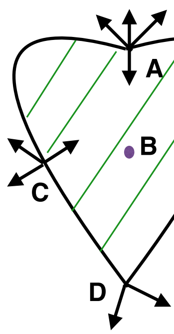
Working sheet 1

1. As shown in the figure, suppose the domain of $f(x, y)$ is the one bounded by the curve(including the boundary). Assume we know that f has local maxima at A, B, C, D.

   (a). Could any of the arrows represent ∇f? (Hint: Understand that ∇f is the fastest increasing direction.) Draw all possible ∇f for all of them.

   Remark: For more information, read materials from online about KKT conditions.

   (b). The condition $g(x, y) = C$ is usually a curve. Now, suppose the boundary in the picture is $g = C$ and the domain of f is $g = C$. Answer the same questions for A, C, D.

2. (a). Consider $f(x, y, z) = 3x^2 + 4y^2 + z^2$ and $g(x, y, z) = 2x + 3y + z = 1$ is the constraint. Find the candidates for extrema of f on the constraint. Is this a minimum point, a maximum point or neither? (Hint: You have a unique candidate. If a max exists, this must be the max; If a min exists, this must be the min. Then, you should argue the existence of them.)

   (b). Consider $f(x, y, z) = 2x + 3y + z$ and $g(x, y, z) = 3x^2 + 4y^2 + z^2 = 1$. Find all candidates for extrema of f on the constraint $g = 1$. Is there a global max? If yes, which one is? Is there a global min? If yes, which one is?

   Soln. (a) We first of all find the candidates. Obviously, $\nabla g \neq 0$. Use Lagrange multiplier, we have

   $$(6x, 8y, 2z) = \nabla f = \lambda \nabla g = \lambda(2, 3, 1)$$

   This means $x = \lambda/3, y = 3\lambda/8, z = \lambda/2$. Plugging them into $2x+3y+z = 1$. You get $\lambda = 24/55$. The point is $(8/55, 9/55, 12/55)$. The problem is how to determine if it's a max or a min.

   You can't use 2nd derivative test. The point you find is with constraint $g = C$ and $\nabla f \neq 0$ in general. The 2nd derivative test usually fails.

Instead, you must look at the function itself. The constraint tells you x, y, z can go to infinity. No matter which direction you go to infinity, you'll always make f go to infinity. Hence, f can't have maximum. Is this point a minimum? Actually, the minimum must exist because as you go far away enough, your f is large enough. There must be a lowest point in the middle. There's only one candidate. The point must be a minimum point.

(b). Similarly, $\nabla g \neq \vec{0}$ on $g = 1$. Using Lagrange multiplier:

$$(2, 3, 1) = \lambda(6x, 8y, 2z)$$

We have $x = 1/(3\lambda), y = 3/(8\lambda), z = 1/(2\lambda)$. Plug this into the constraint:

$$3\frac{1}{9\lambda^2} + 4\frac{9}{64\lambda^2} + \frac{1}{4\lambda^2} = 1$$
$$\lambda^2 = \frac{1}{3} + \frac{9}{16} + \frac{1}{4} = \frac{55}{48} \Rightarrow \lambda = \pm\sqrt{\frac{55}{48}}$$

You therefore get two points:

$$(\frac{\sqrt{48}}{3\sqrt{55}}, \frac{3\sqrt{48}}{8\sqrt{55}}, \frac{\sqrt{48}}{2\sqrt{55}}) = (\frac{4}{\sqrt{165}}, \frac{3\sqrt{3}}{2\sqrt{55}}, \frac{2\sqrt{3}}{\sqrt{55}})$$

and

$$(-\frac{4}{\sqrt{165}}, -\frac{3\sqrt{3}}{2\sqrt{55}}, -\frac{2\sqrt{3}}{\sqrt{55}})$$

Which one is maximum and which one is minimum? Or even, does the max/min exist?

Notice that $3x^2+4y^2+z^2 = 1$. Then, your x, y, z are bounded. They can't go too far. Actually, this is the surface of an ellipsoid. Then, your domain is bounded and closed. There must be a maximum and a minimum. Then, which one is the maximum and which one is the minimum?–The answer is that we plug in values!

$$f(\frac{4}{\sqrt{165}}, \frac{3\sqrt{3}}{2\sqrt{55}}, \frac{2\sqrt{3}}{\sqrt{55}}) > f(-\frac{4}{\sqrt{165}}, -\frac{3\sqrt{3}}{2\sqrt{55}}, -\frac{2\sqrt{3}}{\sqrt{55}})$$

. Since the maximum and the minimum exist, they must be from these two points. Then, you know the first one must be the global max and the latter one must be the global min.

3. As you see in applications, the Lagrange multiplier method is usually performed like this: define the Lagrangian $L(x, y, z, \lambda) = f(x, y, z) + \lambda(g(x, y, z) - C)$. Then the optimization of f with constraint $g = C$ is equivalent to finding the unconstrained critical points of L on the $4D$ space. Convince yourself that this is true. (You see the tricky part

here: one way to get rid of the constraint is to solve the implicit function $z = h(x, y)$ and plug in, then you have an optimization problem in $2D$ space $f(x, y, h(x, y))$; here, we are increasing the dimension! We can solve the unconstraint problem again!)