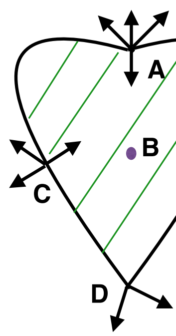
Working sheet 1

1. As shown in the figure, suppose the domain of $f(x,y)$ is the one bounded by the curve(including the boundary). Assume we know that f has local maxima at A, B, C, D.

   (a). Could any of the arrows represent ∇f? (Hint: Understand that ∇f is the fastest increasing direction.) Draw all possible ∇f for all of them.

   Remark: For more information, read materials from online about KKT conditions.

   (b). The condition $g(x,y) = C$ is usually a curve. Now, suppose the boundary in the picture is $g = C$ and the domain of f is $g = C$. Answer the same questions for A, C, D.

2. (a). Consider $f(x,y,z) = 3x^2 + 4y^2 + z^2$ and $g(x,y,z) = 2x + 3y + z = 1$ is the constraint. Find a candidate for extrema of f on the constraint. Is this a minimum point, a maximum point or neither? (Hint: You have a unique candidate. If a max exists, this must be the max; If a min exists, this must be the min. Then, you should argue the existence of them.)

   (b). Consider $f(x,y,z) = 2x + 3y + z$ and $g(x,y,z) = 3x^2 + 4y^2 + z^2 = 1$. Find all candidates for extrema of f on the constraint $g = 1$. Is there a global max? If yes, which one is? Is there a global min? If yes, which one is?

3. As you see in applications, the Lagrange multiplier method is usually performed like this: define the Lagrangian $L(x,y,z,\lambda) = f(x,y,z) + \lambda(g(x,y,z) - C)$. Then the optimization of f with constraint $g = C$ is equivalent to finding the unconstrained critical points of L on the $4D$ space. Convince yourself that this is true. (You see the tricky part here: one way to get rid of the constraint is to solve the implicit function $z = h(x,y)$ and plug in, then you have an optimization problem in $2D$ space $f(x,y,h(x,y))$; here, we are increasing the dimension! We can solve the unconstraint problem again!)