1. (Line integrals–Using parametrization. Two types and the flux integral)

   Formulas: $ds = |\vec{x}'(t)|dt$, $d\vec{x} = \vec{x}'(t)dt$ and $d\vec{x} = \vec{T}ds$ since $\vec{T} = \vec{x}'(t)/|\vec{x}'(t)|$.

   Another one is $\vec{N}ds = \vec{T}ds \times \hat{z} = (dx, dy) \times \hat{z} = (dy, -dx)$ in $2D$.

   (a). Compute the average of the polar angle on $x^2 + y^2 = 4, y \geq 0$.

   The weighted average is $\int_{\mathcal{C}} f ds / \int_{\mathcal{C}} ds$

   In the case here, $f = \theta$. We parametrize the curve with $\vec{x}(t) = (2\cos t, 2\sin t), 0 \leq t \leq \pi$. It's easy to see that $\tan\theta = y(t)/x(t) = \tan t$ and $\theta = t$ actually. Also $ds = |\vec{x}'|dt = 2dt$. Hence, the average is

$$\bar{\theta} = \frac{\int_0^\pi t2dt}{\int_0^\pi 2dt} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

   This makes sense.

   (b). Let $\mathcal{C}$ be $y = \ln x$, $1 \leq x \leq 2$. Compute $\int_{\mathcal{C}} x^2 ds$

   We parametrize the curve as $\vec{x}(t) = (t, \ln t), 1 \leq t \leq 2$. We can compute that $ds = |\vec{x}'|dt = \sqrt{1 + 1/t^2}dt$. Hence, the integral is

$$\int_1^2 t^2\sqrt{1+1/t^2}dt = \int_1^2 t\sqrt{t^2+1}dt = \int_2^5 \sqrt{u}\frac{1}{2}du = \frac{1}{3}u^{3/2}|_{u=2}^{u=5}$$

   (c). Let $\vec{F} = (-2y + 2x, 2x - 2y)$ and $\mathcal{C}$ is $\vec{x}(t) = (t, t^2), 0 \leq t \leq 1$. Compute the work done by $\vec{F}$ along the curve.

   The field is not conservative and the curve is not closed. We have to evaluate directly. The work is $\int_{\mathcal{C}} \vec{F} \cdot d\vec{x}$

   Under the parametrization, $\vec{F} = (-2t^2 + 2t, 2t - 2t^2)$ and $d\vec{x} = (1, 2t)dt$. Hence, the integral is

$$\int_0^1 (1 * (-2t^2 + 2t) + 2t(2t - 2t^2))dt = \int_0^1 (2t + 2t^2 - 4t^3)dt = 2/3$$

   (d). Let $\mathcal{C}$ be the line segment from $(1, 2)$ to $(-1, 2)$. Find the rate at which the amount of fluid flows across this curve where the velocity field is $\vec{v} = 4xy\hat{i} - y^2\hat{j}$.

   Parametrize the curve: $\vec{x}(t) = (1 - 2t, 2), 0 \leq t \leq 1$. The integral is $\int_{\mathcal{C}} \vec{v} \cdot \vec{N}ds$. The curve is not closed and we have to compute directly.

   You may want to find $\vec{N}$ and ds respectively. If you do this, $\vec{x}' = (-2, 0)$ and $\vec{T} = (-1, 0)$. Hence $\vec{N} = (0, 1)$ and $ds = |\vec{x}'|dt = 2dt$. However, we can use the formula directly $\vec{N}ds = (dy, -dx) = (0dt, -(-2dt)) = (0, 2)dt$. Anyway, the integral is

$$\int_0^1 \begin{pmatrix} 4(1-2t) * 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} dt = -8$$

   This means the flow is actually downward.

2. (Line integrals of conservative fields–the fundamental theorem)

(a). Is the field $\vec{F} = (\sin(y^2) + 4x^3y, 2xy\cos(y^2) + x^4)$ conservative? If yes, find the potential ϕ so that $\vec{F} = -\nabla\phi$(note in our current textbook, there's a negative sign in the front of gradient.)

We check that $P_y = 2y\cos(y^2) + 4x^3$ and $Q_x = 2y\cos(y^2) + 4x^3$. They are equal and the domain is simply connected. Then, yes. To find the potential, we find f first.

$$f = \int P dx = x\sin(y^2) + x^4y + g(y)$$

Using the fact $f_y = Q$, we figure out $g' = 0$ and hence $f = x\sin(y^2) + x^4y + C$. One possible $\phi = -x\sin(y^2) - x^4y$

(b). Let $\vec{F} = (2y+2x, 2x-2y)$ and $\mathcal{C}$ is $\vec{x}(t) = (t, t^2), 0 \le t \le 1$. Compute the work done by $\vec{F}$ along the curve.

Notice that the field is conservative and we find $f = x^2 + 2xy - y^2 + C$. The starting point is $(0,0)$ while the end point is $(1,1)$. Hence

$$\int \nabla f \cdot d\vec{x} = f(1,1) - f(0,0) = 2$$

(c). Let $\vec{F} = (2x+2y, 2x+2y, z)$. Is this field conservative? Let $\mathcal{C}$ be the line segment from $(1,1,0)$ to $(1,2,2)$. Find the line integral $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}ds$

We check that $\nabla\times\vec{F} = 0$. It's conservative. If you observe, you may guess out the function quickly $f = x^2 + 2xy + y^2 + \frac{1}{2}z^2 + C$. If you can't, you may just integrate:

$$f = \int (2x+2y)dx = x^2 + 2xy + g(y,z)$$

Then, $f_y = 2x + g_y = 2x + 2y$ and hence $g_y = 2y$.

$$g = \int 2y dy = y^2 + h(z)$$

Finally, $f_z = g_z = h'(z) = z$. You solve $h(z) = z^2/2 + C$

Noticing $\vec{T}ds = d\vec{x}$ and using the fundamental theorem, the answer is

$$f(1,2,2) - f(1,1,0) = (1+4+4+2) - (1+2+1+0) = 7$$

3. (Line integrals on closed curve–Green's theorem. Two versions)

(a). Let $\mathcal{C}$ be the boundary of the region $0 \le y \le 1, y^2 \le x \le 1$ with counterclockwise orientation. Compute $\oint_{\mathcal{C}} y^2\sin(x^2)dx$

Applying Green's, we have

$$\iint_D -2y\sin(x^2)dA = -2\int_0^1\int_{y^2}^1 y\sin(x^2)dxdy$$

Change the order of integration

$$-2\int_0^1\int_0^{\sqrt{x}} y\sin(x^2)dydx = -\int_0^1 x\sin(x^2)dx = \frac{1}{2}\cos 1 - \frac{1}{2}$$

(b). Given $\vec{F} = (x^2y, -xy^2)$ and $\mathcal{C}$ is the boundary of the unit circle oriented counterclockwisely. Compute the circulation of $\vec{F}$ on the curve. (Circulation is $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{x}$)

The circulation equals

$$\oint_{\mathcal{C}} x^2ydx - xy^2dy = \iint_D (-x^2-y^2)dA = -\int_0^{2\pi}\int_0^1 r^2 rdrd\theta = -\frac{\pi}{2}$$

(c). Let $\vec{v} = (2x^3, -y^3)$ and $\mathcal{C}$ be the circle $x^2+y^2=4$ oriented counterclockwisely. Compute the outer flux of $\vec{v}$ on the curve.

$$\begin{aligned}\oint_{\mathcal{C}} \vec{v}\cdot\vec{N}ds = \iint_D \nabla\cdot\vec{v}dA &= \iint_D (6x^2-3y^2)dA \\ &= \int_0^{2\pi}\int_0^2 (6*r^2\cos^2\theta - 3*r^2\sin^2\theta)rdrd\theta\end{aligned}$$

We integrate on θ first using double angle formula and have

$$\pi\int_0^2 (6*r^2 - 3*r^2)rdr = \pi\frac{3}{4}*2^4 = 12\pi$$

(d). Compute $\oint_{\mathcal{C}} xdy$ (i). Along the boundary of ellipse $x^2/4+y^2=1$ (ii). Along a ∞ shaped curve with the right loop oriented counterclockwisely, assuming the right loop encloses area 3 and the left loop encloses area 2.

(i). We compute the line integral directly. $\vec{x}(t) = (2\cos t, \sin t), 0 \le t < 2\pi$. Notice that t is no longer the polar angle opposed to the circle case. Then, the integral is $\int_0^{2\pi} 2\cos t\cos tdt = \int_0^{2\pi}(1+\cos(2t))dt = 2\pi$. Actually, by Green's, the integral is $\iint dxdy = Area$.

(ii). Applying Green's, we have $\iint_D dA$. However, the loop is counterclockwise on the right but clockwise on the left half. So the answer is $Area(right) - Area(left) = 3-2 = 1$

(e*) Let $\mathcal{C}$ be any closed in the plane that encloses the origin. Let $\vec{v} = (-y/(x^2+y^2), x/(x^2+y^2))$. Compute $\oint_{\mathcal{C}} \vec{v} \cdot d\vec{x}$.

The idea is to isolate the singular point $(0,0)$ with a small circle with radius r because we can't apply Green's directly. Let $\mathcal{C}_1$ be the boundary of the small circle. Then, noticing $P_y = Q_x$ for $\vec{v}$, we have

$$\int_{\mathcal{C}} \vec{v} \cdot d\vec{x} = \int_{\mathcal{C}_1} \vec{v} \cdot d\vec{x}$$

On this small circle, $x = r\cos t, y = r\sin t$ or $\vec{x}(t) = (r\cos t, r\sin t), 0 \le t < 2\pi$. Using the parametrization, we can compute the line integral directly. The answer is 2π

4. (Surface Integrals–Basic definition and parametrization)

(a). Parametrize the surfaces and compute $d\vec{S} = \vec{N}dA$, dA for (i) $0 \le x, y \le 3, z = 1$. (ii). $z = xy, 1 \le x \le 2, 1 \le y \le 3$. (iii). $z = \sqrt{x^2+y^2}, 1 \le z \le 2$

(i). We let $u = x, v = y$ and we see that $z = 1$ always. Hence, $\vec{x}(u,v) = u\hat{i} + v\hat{j} + \hat{k} = (u, v, 1)$ with $0 \le u, v \le 3$. $\vec{N}dA = \vec{x}_u \times \vec{x}_v dudv = \hat{k}dudv$. Hence, $dA = |\vec{N}dA| = |\hat{z}dudv| = dudv$

(ii). Let $x = u, y = v$. Then, $z = uv$. Then, $\vec{x}(u,v) = (u, v, uv)$ with $1 \le u \le 2, 1 \le v \le 3$. We compute that $\vec{N}dA = (1,0,v) \times (0,1,u)dudv = (-v,-u,1)dudv$. Then, $dA = |(-v,-u,1)|dudv = \sqrt{1+u^2+v^2}dudv$

(iii). You can use Cartesian as well. Here, a better choice is to use cylindrical coordinates. Then, we see that $x = r\cos\theta, y = r\sin\theta, z = r$. Hence, we have $\vec{x}(r,\theta) = (r\cos\theta, r\sin\theta, r)$ with $1 \le r \le 2, 0 \le \theta < 2\pi$. We then, compute that

$$\vec{N}dA = (\cos\theta, \sin\theta, 1) \times (-r\sin\theta, r\cos\theta, 0)drd\theta = (-r\cos\theta, -r\sin\theta, r)drd\theta$$

Hence $dA = r\sqrt{1+1}drd\theta = \sqrt{2}rdrd\theta$

Compare this with flat plane in cylindrical coordinates, or polar coordinates. In polar, $\vec{x} = (r\cos\theta, r\sin\theta, z_0)$ where z_0 is a constant. You get $dA = rdrd\theta$, which is different from our dA here.

(b). Let $\mathcal{S}$ be the surface $z = \sin(xy)$ for $0 \le x, y \le \pi/2$. Set up the integral for $\iint_{\mathcal{S}} (x+z)dA$

$\vec{x}(u,v) = (u, v, \sin(uv))$.

$$dA = |\vec{x}_u \times \vec{x}_v|dudv = \sqrt{v^2\cos^2(uv) + u^2\cos^2(uv) + 1}dudv$$

The integral is

$$\int_0^{\pi/2}\int_0^{\pi/2} (u + \sin(uv))\sqrt{v^2\cos^2(uv) + u^2\cos^2(uv) + 1}dudv$$

(c*). Consider the surface determined by $F(x,y,z) = 1, 1 \le z \le 2$ where $F(x,y,z) = x^2+y^2-z^2$. Set up the integral $\iint_{\mathcal{S}} \vec{v}\cdot\vec{N}dA$ where $\vec{v} = (x,1,0)$.

We parametrize the surface as $\vec{x}(x,y) = (x,y,z(x,y))$. Using implicit differentiation, we have

$$\vec{x}_x = (1,0,\frac{x}{z}), \vec{x}_y = (0,1,\frac{y}{z})$$

We have

$$\vec{N}dA = \vec{x}_x \times \vec{x}_y dxdy = (-x/z,-y/z,1)dxdy$$

The integral is

$$\iint_D (-\frac{x^2}{z} - \frac{y}{z})dxdy$$

Further, we solve $z = \sqrt{x^2+y^2-1}$ and we can plug in. The integration region is $1^2+1 \le x^2+y^2 \le 2^2+1$ or $2 \le x^2+y^2 \le 5$.

Using polar coordinates, the integral is

$$\int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{5}} (-\frac{r^2\cos^2\theta}{\sqrt{r^2-1}} - \frac{r\sin\theta}{\sqrt{r^2-1}})rdrd\theta$$

5. (Changing line integrals to flux integrals(surface version)-Stokes Theorem)

(a). Let $\mathcal{C}$ be the intersection of $x^2+y^2=4$ with $x+y+z=8$. Evaluate $\oint_{\mathcal{C}} (x+y)dx+(y+z)dy+(z+x)dz$ in two ways, where $\mathcal{C}$ is counterclockwise when viewed above.

This is the circulation $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{x}$ where $\vec{F} = (x+y,y+z,z+x)$.

The first way is to evaluate this directly by parametrization: Let $x = 2\cos t, y = 2\sin t$. Then, z can be determined from the equation of the plane: $z = 8-x-y = 8-2\sin t-2\cos t$. Hence, the curve can be parametrized as $\vec{x}(t) = (2\cos t, 2\sin t, 8-2\cos t-2\sin t)$. Also, $d\vec{x} = (-2\sin t, 2\cos t, 2\sin t - 2\cos t)dt$. On this curve, the force is $\vec{F} = (2\cos t + 2\sin t, 8-2\cos t, 8-2\sin t)$ .The line integral is thus

$$\int_0^{2\pi} (2\sin t+2\cos t)(-2\sin t)dt+(8-2\cos t)2\cos tdt+(8-2\sin t)(2\sin t-2\cos t)dt$$
$$= -4\pi-4\pi-4\pi = -12\pi$$

The second way is to use Stokes Theorem. We pick the surface to be the plane $z = 8-x-y$ inside the cylinder. The surface can thus be parametrized as $\vec{x}(x,y) = (x,y,8-x-y)$ with $x^2+y^2 \le 4$

We compute the curl $\nabla \times \vec{F} = (-1, -1, -1)$. We then compute

$$\vec{N}dA = (1, 0, -1) \times (0, 1, -1)dxdy = (1, 1, 1)dxdy$$

Hence, we have

$$\iint_D (-3)dxdy = \int_0^{2\pi} \int_0^2 (-3)rdrd\theta = -3 * 4\pi$$

(b). Use Stokes' Theorem to compute the circulation of $\vec{F} = (y^2 + z^2)\hat{\imath} + (x^2 + z^2)\hat{\jmath} + (x^2 + y^2)\hat{k}$ along the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

$\nabla \times \vec{F} = (2y - 2z, 2z - 2x, 2x - 2y)$. We use the surface as the triangle. Then, we parametrize the surface as $\vec{x}(x, y) = (x, y, 1 - x - y)$. We have $\vec{N}dA = (1, 1, 1)dxdy$. Hence the integral is

$$\iint_D (2y - 2z + 2z - 2x + 2x - 2y)dxdy = 0$$

(c). Let S be the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$ with normal pointing away from origin. Let $\vec{v} = (-y, x^3 + xy^2, xyz)$. Compute the flux:

$$\iint_{\mathcal{S}} curl(\vec{v}) \cdot \vec{N}dA$$

First of all, using Stokes, this is equal to

$$\oint_{\mathcal{C}} \vec{v} \cdot d\vec{x}$$

where $\mathcal{C}$ is the circle $x^2 + y^2 = 9, z = 0$. Here, one may parametrize the curve and get

$$\vec{x} = (3\cos t, 3\sin t, 0), 0 \leq t < 2\pi$$

and reduce the integral to

$$\int_0^{2\pi} (-3\sin t, 27\cos^3 t + 27\cos t \sin^2 t, 0) \cdot (-3\sin t, 3\cos t, 0)dt$$

Another smarter way is to notice $dz = 0$ on the circle and thus the integral is actually

$$\oint_{\mathcal{C}} -ydx + (x^3 + xy^2)dy$$

in xy plane. Then, applying Green's Theorem, this is

$$\iint_D (1 + 3x^2 + y^2)dA$$

where D is the disk. This can be evaluated using polar coordinates.

(d). Let $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,-1)$. Consider the closed curve $\mathcal{C} = AB + BC + CA$. Let $\vec{F} = (-y, x, z)$. Compute the line integral $\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}ds$.

The integral is just $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{x}$. The curve is counterclockwise when viewed above. Hence, we can apply Stokes' Theorem and have:

$$\oint_{\mathcal{C}} \vec{F}\cdot d\vec{x} = \iint_{\mathcal{S}} \nabla\times\vec{F}\cdot\vec{N}dA$$

, the flux of curl. Here $\mathcal{S}$ can be picked as the plane determined by the three points.

It's easy to compute that $\nabla\times\vec{F} = (0,0,2)$. Now we need to determine the plane:

You can determine the normal $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC}$ and use $AP\cdot\vec{n} = 0$ to figure out. However, according to the three points, it's easy to find the plane is $x/1 + y/2 + z/(-1) = 1$. In general, the plane that passes $(a,0,0), (0,b,0), (0,0,c)$ is $x/a + y/b + z/c = 1$. Anyhow, the surface is parametrized as

$$\vec{x}(x,y) = (x, y, x + y/2 - 1), x + y/2 \le 1, x \ge 0, y \ge 0$$

$\vec{N}dA = (-1, -1/2, 1)dxdy$. Hence the integral is

$$\int_0^1\int_0^{2(1-x)} 2dydx = \int_0^1 4(1-x)dx = 2$$

6. (Reducing flux integral on closed surfaces to volume integrals-Divergence Theorem)

   (a). Let R be the region inside the sphere $x^2+y^2+z^2 = 4$ and above xy plane. Let S be the boundary of this region which is thus closed. Compute the flux $\oiint_S \vec{v}\cdot\vec{N}dA$ where $\vec{v} = (xy^2, x^2y + y^3/3, x^2z)$.

   Applying divergence theorem, we have

$$\iiint_D \nabla\cdot\vec{v}dV = \iiint_D (y^2 + (x^2+y^2) + x^2)dV$$

   This is good for spherical coordinates. $2(x^2+y^2) = 2\rho^2\sin^2\phi$, $dV = \rho^2\sin\phi d\rho d\phi d\theta$ and $0\le\rho\le 2, 0\le\theta<2\pi, 0\le\phi\le\pi/2$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 2\rho^2\sin^2\phi\rho^2\sin\phi d\rho d\phi d\theta$$

For $\sin^3\phi$, just write it as $(1-\cos^2\phi)\sin\phi$ and do sub $u=\cos\phi$.

(b). Let D be the region $x^2+y^2\le 4, 0\le z\le 3$. Compute the flux of $\vec{v}=(-y,x,z)$ through the boundary of D. Draw the field and explain intuitively why the flux is positive.

Applying the divergence,

$$\iiint_D (0+0+1)dV = \iiint_D dV$$

The region is good for cylindrical coordinates. $0\le r\le 2, 0\le\theta<2\pi, 0\le z\le 3, dV = rdrd\theta dz$. Answer is 12π

(c). Let $\vec{v}=(y^2, xyz, xz^2)$. Compute the rate at which the fluid flows out of the cube $0\le x,y,z\le 1$.

We need to evaluate $\oiint_S \vec{v}\cdot\vec{N}dA$. Applying divergence theorem,

$$\iiint_D (0+xz+2xz)dV = \int_0^1\int_0^1\int_0^1 3xzdxdydz = 3/4$$

(d). Consider $\mathcal{S}_1 : z=\sqrt{x^2+y^2}$ and $\mathcal{S}_2 : z = x^2+y^2$. These two surfaces enclose a region $\mathcal{R}$. The boundary of this region is $\mathcal{S}$ with outer normal $\vec{N}$. Compute the flux integral $\iint_{\mathcal{S}} \vec{F}\cdot\vec{N}dA$ where $\vec{F}=(y,x,z^2/2)$.

The surface is closed. We just apply the Divergence theorem and have

$$\iiint_{\mathcal{R}} \nabla\cdot\vec{F}dV = \iiint_{\mathcal{R}} zdV$$

The region is actually above $z=x^2+y^2$ and below $z=\sqrt{x^2+y^2}$. The intersection of them is when $x^2+y^2=1$ or $x=y=0$. Anyway, this region can be written conveniently in cylindrical coordinate, which is $0\le r\le 1$, $0\le\theta<2\pi$, $r^2\le z\le r$. Hence the integral is

$$\int_0^1\int_0^{2\pi}\int_{r^2}^{r} z[rdzd\theta dr] = 2\pi\int_0^1 (\frac{1}{2}r^3-\frac{1}{2}r^5)dr = \pi/12$$

7. (Curl, Divergence etc)

Let $\vec{v}=(xy^2, x^2y+y^3/3, x^2z)$. Compute the curl $curl(\vec{v})=\nabla\times\vec{v}$ and the divergence $div(\vec{v})=\nabla\cdot\vec{v}$.

Divergence is easy:

$$\nabla\cdot\vec{v} = (xy^2)_x + (x^2y+y^3/3)_y + (x^2z)_z = 2x^2+2y^2$$

Curl is

$$\begin{vmatrix} \vec{i} & \partial_x & xy^2 \\ \vec{j} & \partial_y & x^2y + y^3/3 \\ \vec{k} & \partial_z & x^2z \end{vmatrix}$$

$$= \vec{i}[(x^2z)_y - (x^2y+y^3/3)_z] - \vec{j}[(x^2z)_x - (xy^2)_z] + \vec{k}[(x^2y+y^3/3)_x - (xy^2)_y]$$
$$= (0, -2xz, 0)$$

8. (Tangent planes, Implicit differentiation, chain rule)

(a). Consider the implicit functions defined by $xy - yz + e^{xz} = 3$. Compute $\partial z/\partial x$.

If you forget the formula, use $F(x, y, z(x, y)) = C$ to recover. Taking derivative on x, you have $F_x + F_z \frac{\partial z}{\partial x} = 0$. The derivative is

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + ze^{xz}}{-y + xe^{xz}}$$

(b). Find the tangent plane of the surface defined in (a) at the point $(0, -1, z_0)$.

We first solve z_0. Plugging the values, $0 + z_0 + 1 = 3$ and hence $z_0 = 2$. To compute the tangent plane of the level set, we compute the normal vector $\nabla F = (y + ze^{xz}, x - z, -y + xe^{zx}) = (1, -2, 1)$. The tangent plane is

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x - 0 \\ y + 1 \\ z - 2 \end{pmatrix} = 0$$

(c). Compute the linear approximation of $z = 2^{x-y}$ at $(1, 1)$, and the tangent line of the level set passing through $(1, 1)$.

Let $f(x, y) = 2^{x-y}$. We have $f(1, 1) = 1$, $f_x = 2^{x-y} \ln 2$ and $f_x(1, 1) = \ln 2$. Similarly, $f_y(1, 1) = -\ln 2$. Hence the linear approximation is

$$f(x, y) \approx 1 + \ln 2(x - 1) - \ln 2(y - 1)$$

The graph of this function is the tangent plane of the graph. Also you should be able to compute the second order Taylor expansion.

The tangent line to the level set is

$$(\ln 2)(x - 1) - (\ln 2)(y - 1) = 0$$

(d). If $\nabla f(1,2,3) = (-2,3,5)$. Let $g(x,y) = f(xy, x+y, y^2+2x)$. Compute $g_x(1,1)$ and $g_y(1,1)$

Use the chain rule. We assume the function $f(u,v,w)$. Then, we have

$$g_x(x,y) = f_u u_x + f_v v_x + f_w w_x = f_u y + f_v * 1 + f_w * 2$$

Plugging in $(1,1)$, we have

$$g_x(1,1) = f_u(1,2,3)*1 + f_v(1,2,3)*1 + f_w(1,2,3)*2 = -2+3+10 = 11$$

Computing $g_y(1,1)$ is similar. The answer is 11.

9. (Volume integrals in spherical coordinates; cylindrical coordinates)

(a). Find the moment of inertia about z-axis of the ball $x^2+y^2+z^2 \le 1$ with density $\mu(x,y,z) = z^2$

The moment of inertia about z axis is

$$I_z = \iiint_D (x^2+y^2)\mu(x,y,z)dV$$

This problem is good for spherical. $0 \le \rho \le 1, 0 \le \theta < 2\pi, 0 \le \phi \le \pi$ and $\mu = \rho^2 \cos^2\phi$. $x^2+y^2 = \rho^2 \sin^2\phi$. Further, $dV = \rho^2 \sin\phi d\rho d\theta d\phi$

Then, the integral is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin^2\phi \rho^2 \cos^2\phi \rho^2 \sin\phi d\rho d\phi d\theta = \frac{2\pi}{7}\int_0^{\pi} \sin^3\phi \cos^2\phi d\phi$$

This integral can be evaluated by writing $\sin^3\phi = (1-\cos^2\phi)\sin\phi$ and doing $u = \cos\phi$. You can finish it

(b). Find the center of mass of the region inside $x^2+y^2 \ge 1, x^2+y^2 \le 4$, bounded by $z = x^2+y^2$ and xy plane with unit density.

The domain is symmetric in x and $\mu(x,y,z) = 1 = \mu(-x,y,z)$. We know $\bar{x} = 0$ by symmetry. Similarly, $\bar{y} = 0$. We only have to compute $\bar{z}$ which is

$$\bar{z} = \frac{\iiint z\mu dV}{\iiint \mu dV}$$

The domain is good for cylindrical. In cylindrical, we have $1 \le r \le 2, 0 \le \theta < 2\pi, 0 \le z \le r^2$ and $\mu = 1$. Further, $dV = r dr d\theta dz$. Hence, we have

$$\bar{z} = \frac{\int_1^2\int_0^{2\pi}\int_0^{r^2} zr dz d\theta dr}{\int_1^2\int_0^{2\pi}\int_0^{r^2} r dz d\theta dr} = \frac{\pi\int_1^2 r^5 dr}{2\pi\int_1^2 r^3 dr} = \frac{63}{45}$$

The center is $(0,0,63/45)$

10. (2nd derivative test; Lagrange multiplier)

(a). Consider $f(x, y) = x^3 + y^3 - 3xy$. Find all critical points on $\mathbb{R}^2$ and classify them.

$f_x = 3x^2 - 3y = 0$ and $f_y = 3y^2 - 3x = 0$. Hence, $x^4 - x = 0$ and we have $x = 0, x = 1$. Using $y = x^2$, we have $(0, 0), (1, 1)$.

$f_{xx} = 6x, f_{xy} = -3, f_{yy} = 6y$

At $(0, 0)$, $f_{xx}f_{yy} - f_{xy}^2 < 0$, the point is a saddle point; at $(1, 1)$, the three numbers are $6, -3, 6$ respectively and $f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 > 0$, further $f_{xx} > 0$ and the point is a local min.

(b). Find the minimum surface area of a rectangular box without bottom, provided the volume is V.

Let the dimensions be x, y, z. $f(x, y) = xy + 2xz + 2yz$ with the constraint $g(x, y, z) = xyz = V$. Then

$$\begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2x + 2y &= \lambda xy \end{aligned}$$

Now, you have $x(y + 2z) = y(x + 2z) = z(2x + 2y)$. Then, $x = y$, $x = 2z$. Hence $x^2 * x/2 = V$ or $x = \sqrt[3]{2V}$. Hence y, z can be found.

There must be a minimum value and we only have one candidate. This should be the point. Plugging the values and the area is ...