

1. (Line integrals—Using parametrization. Two types and the flux integral)

Formulas: $ds = |\vec{x}'(t)|dt$, $d\vec{x} = \vec{x}'(t)dt$ and $d\vec{x} = \vec{T}ds$ since $\vec{T} = \vec{x}'(t)/|\vec{x}'(t)|$.

Another one is $\vec{N}ds = \vec{T}ds \times \hat{z} = (dx, dy) \times \hat{z} = (dy, -dx)$ in $2D$.

 - (a). Compute the average of the polar angle on $x^2 + y^2 = 4, y \geq 0$.
 - (b). Let C be $y = \ln x, 1 \leq x \leq 2$. Compute $\int_C x^2 ds$
 - (c). Let $\vec{F} = (-2y + 2x, 2x - 2y)$ and C is $\vec{x}(t) = (t, t^2), 0 \leq t \leq 1$. Compute the work done by \vec{F} along the curve.
 - (d). Let C be the line segment from $(1, 2)$ to $(-1, 2)$. Find the rate at which the amount of fluid flows across this curve where the velocity field is $\vec{v} = 4xy\hat{i} - y^2\hat{j}$.
2. (Line integrals of conservative fields—the fundamental theorem)
 - (a). Is the field $\vec{F} = (\sin(y^2) + 4x^3y, 2xy \cos(y^2) + x^4)$ conservative? If yes, find the potential ϕ so that $\vec{F} = -\nabla\phi$ (note in our current textbook, there's a negative sign in the front of gradient.)
 - (b). Let $\vec{F} = (2y + 2x, 2x - 2y)$ and C is $\vec{x}(t) = (t, t^2), 0 \leq t \leq 1$. Compute the work done by \vec{F} along the curve.
 - (c). Let $\vec{F} = (2x + 2y, 2x + 2y, z)$. Is this field conservative? Let C be the line segment from $(1, 1, 0)$ to $(1, 2, 2)$. Find the line integral $\int_C \vec{F} \cdot \vec{T} ds$
3. (Line integrals on closed curve—Green's theorem. Two versions)
 - (a). Let C be the boundary of the region $0 \leq y \leq 1, y^2 \leq x \leq 1$ with counterclockwise orientation. Compute $\oint_C y^2 \sin(x^2) dx$
 - (b). Given $\vec{F} = (x^2y, -xy^2)$ and C is the boundary of the unit circle oriented counterclockwisely. Compute the circulation of \vec{F} on the curve. (Circulation is $\oint_C \vec{F} \cdot d\vec{x}$)
 - (c). Let $\vec{v} = (2x^3, -y^3)$ and C be the circle $x^2 + y^2 = 4$ oriented counterclockwisely. Compute the outer flux of \vec{v} on the curve.
 - (d). Compute $\oint_C x dy$ (i). Along the boundary of ellipse $x^2/4 + y^2 = 1$ (ii). Along a ∞ shaped curve with the right loop oriented counterclockwisely, assuming the right loop encloses area 3 and the left loop encloses area 2.
 - (e*) Let C be any closed in the plane that encloses the origin. Let $\vec{v} = (-y/(x^2 + y^2), x/(x^2 + y^2))$. Compute $\oint_C \vec{v} \cdot d\vec{x}$.
4. (Surface Integrals—Basic definition and parametrization)
 - (a). Parametrize the surfaces and compute $d\vec{S} = \vec{N}dA, dA$ for (i) $0 \leq x, y \leq 3, z = 1$. (ii). $z = xy, 1 \leq x \leq 2, 1 \leq y \leq 3$. (iii). $z = \sqrt{x^2 + y^2}, 1 \leq z \leq 2$
 - (b). Let S be the surface $z = \sin(xy)$ for $0 \leq x, y \leq \pi/2$. Set up the integral for $\iint_S (x + z) dA$

- (c*). Consider the surface determined by $F(x, y, z) = 1, 1 \leq z \leq 2$ where $F(x, y, z) = x^2 + y^2 - z^2$. Set up the integral $\iint_S \vec{v} \cdot \vec{N} dA$ where $\vec{v} = (x, 1, 0)$.
5. (Changing line integrals to flux integrals(surface version)-Stokes Theorem)
- (a). Let \mathcal{C} be the intersection of $x^2 + y^2 = 4$ with $x + y + z = 8$. Evaluate $\oint_{\mathcal{C}} (x+y)dx + (y+z)dy + (z+x)dz$ in two ways, where \mathcal{C} is counterclockwise when viewed above.
- (b). Use Stokes' Theorem to compute the circulation of $\vec{F} = (y^2 + z^2)\hat{i} + (x^2 + z^2)\hat{j} + (x^2 + y^2)\hat{k}$ along the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.
- (c). Let S be the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$ with normal pointing away from origin. Let $\vec{v} = (-y, x^3 + xy^2, xyz)$. Compute the flux:

$$\iint_S \text{curl}(\vec{v}) \cdot \vec{N} dA$$

- (d). Let $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, -1)$. Consider the closed curve $\mathcal{C} = AB + BC + CA$. Let $\vec{F} = (-y, x, z)$. Compute the line integral $\oint_{\mathcal{C}} \vec{F} \cdot \vec{T} ds$.
6. (Reducing flux integral on closed surfaces to volume integrals-Divergence Theorem)
- (a). Let R be the region inside the sphere $x^2 + y^2 + z^2 = 4$ and above xy plane. Let S be the boundary of this region which is thus closed. Compute the flux $\iint_S \vec{v} \cdot \vec{N} dA$ where $\vec{v} = (xy^2, x^2y + y^3/3, x^2z)$.
- (b). Let D be the region $x^2 + y^2 \leq 4, 0 \leq z \leq 3$. Compute the flux of $\vec{v} = (-y, x, z)$ through the boundary of D . Draw the field and explain intuitively why the flux is positive.
- (c). Let $\vec{v} = (y^2, xyz, xz^2)$. Compute the rate at which the fluid flows out of the cube $0 \leq x, y, z \leq 1$.
- (d). Consider $\mathcal{S}_1 : z = \sqrt{x^2 + y^2}$ and $\mathcal{S}_2 : z = x^2 + y^2$. These two surfaces enclose a region \mathcal{R} . The boundary of this region is \mathcal{S} with outer normal \vec{N} . Compute the flux integral $\iint_S \vec{F} \cdot \vec{N} dA$ where $\vec{F} = (y, x, z^2/2)$.
7. (Curl, Divergence etc)
- Let $\vec{v} = (xy^2, x^2y + y^3/3, x^2z)$. Compute the curl $\text{curl}(\vec{v}) = \nabla \times \vec{v}$ and the divergence $\text{div}(\vec{v}) = \nabla \cdot \vec{v}$.
8. (Tangent planes, Implicit differentiation, chain rule)
- (a). Consider the implicit functions defined by $xy - yz + e^{xz} = 3$. Compute $\partial z / \partial x$.
- (b). Find the tangent plane of the surface defined in (a) at the point $(0, -1, z_0)$.

- (c). Compute the linear approximation of $z = 2^{x-y}$ at $(1, 1)$, and the tangent line of the level set passing through $(1, 1)$.
- (d). If $\nabla f(1, 2, 3) = (-2, 3, 5)$. Let $g(x, y) = f(xy, x + y, y^2 + 2x)$. Compute $g_x(1, 1)$ and $g_y(1, 1)$
9. (Volume integrals in spherical coordinates; cylindrical coordinates)
- (a). Find the moment of inertia about z -axis of the ball $x^2 + y^2 + z^2 \leq 1$ with density $\mu(x, y, z) = z^2$
- (b). Find the center of mass of the region inside $x^2 + y^2 \geq 1, x^2 + y^2 \leq 4$, bounded by $z = x^2 + y^2$ and xy plane with unit density.
10. (2nd derivative test; Lagrange multiplier)
- (a). Consider $f(x, y) = x^3 + y^3 - 3xy$. Find all critical points on \mathbb{R}^2 and classify them.
- (b). Find the minimum surface area of a rectangular box without bottom, provided the volume is V .