Review Working sheet 2

1. (Critical points, Taylor expansion, and 2nd derivative test)

   (a) Find the Taylor expansion of $f(x,y) = 2^{x-y^2}$ at $(1,1)$ up to second order.

   $f_x = 2^{x-y^2}\ln 2$, $f_y = 2^{x-y^2}\ln 2(-2y)$, $f_{xx} = 2^{x-y^2}(\ln 2)^2$, $f_{xy} = 2^{x-y^2}(\ln 2)^2(-2y)$ and $f_{yy} = 2^{x-y^2}(\ln 2)^2 4y^2 - 2(\ln 2)2^{x-y^2}$

   $f(1,1) = 2^0 = 1$, $f_x(1,1) = \ln 2$, $f_y(1,1) = -2\ln 2$, $f_{xx}(1,1) = (\ln 2)^2$, $f_{xy}(1,1) = -2(\ln 2)^2$, and $f_{yy}(1,1) = 4(\ln 2)^2 - 2\ln 2$

   The Taylor expansion($f \approx f(1,1)+f_x(1,1)\Delta x+f_y(1,1)\Delta y+\frac{1}{2}f_{xx}(1,1)\Delta x^2+ f_{xy}(1,1)\Delta x\Delta y + \frac{1}{2}f_{yy}(1,1)\Delta y^2$) is

   $$f(x,y) \approx 1+\ln 2(x-1)-2\ln 2(y-1)+\frac{1}{2}(\ln 2)^2(x-1)^2-2(\ln 2)^2(x-1)(y-1)+(2(\ln 2)^2-\ln 2)(y-1)^2$$

   (b) Find all critical points and apply the 2nd derivative test for the following:

   $$f(x,y) = 8x^4 + y^4 - xy^2$$

   $f_x = 32x^3 - y^2 = 0$, $f_y = 4y^3 - 2xy = 0$

   The second equation: $y = 0$ or $x = 2y^2$. If $y = 0$, $x = 0$ and we have a critical point $(0,0)$. If $x = 2y^2$, we have $32 * 8 * y^6 - y^2 = 0$. Then, $y = 0$ or $y = \pm 1/(256)^{1/4} = \pm 1/4$. This gives $(0,0)$, $(1/8, 1/4)$ and $(1/8, -1/4)$. In all, we have $(0,0)$, $(1/8, \pm 1/4)$

   $f_{xx} = 96x^2$, $f_{xy} = -2y$, $f_{yy} = 12y^2 - 2x$.

   At $(0,0)$, the three numbers are $0,0,0$ respectively. The 2nd derivative test is inconclusive as the form is semidefinite. However, if you look around $(0,0)$, you'll see both negative value and positive value since around origin, the main term is $-xy^2$. Hence, this should be kind of saddle.

   At $(1/8, 1/4)$, the three numbers are $3/2, -1/2, 1/2$. Then, we have $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$, this is a local min.

   At $(1/8, -1/4)$, the three numbers are $3/2, 1/2, 1/2$. $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$. It's still a local min.

2. (Optimization with constraint, Lagrange multiplier)

   (a). Find all the points on the surface $xy - z^2 + 1 = 0$ that are closest to the origin.

   Let (x,y,z) be a point on the surface. Then the distance is $\sqrt{x^2+y^2+z^2}$. We choose $f(x,y,z) = x^2+y^2+z^2$ to minimize with constraint $g(x,y,z) = xy - z^2 + 1 = 0$

   $\nabla g \neq \vec{0}$ on the surface, because otherwise the constraint can't be satisfied. Hence, we have

   $$\nabla f = \lambda \nabla g \quad g = 0$$

We have equations $2x = \lambda y$, $2y = \lambda x$, $2z = \lambda(-2z)$, $xy - z^2 + 1 = 0$

By the first equation, $x = \lambda y/2$. Plugging this into the second equation, we have $2y = \lambda^2 y/2$. We have $y(2 - \lambda^2/2) = 0$.

If $y = 0$, then $x = 0$. The first two equations are dealt with. Plugging them into the last equation, we get $z = \pm 1$. The third equation tells us that $\lambda = -1$. Hence, $(0, 0, \pm 1)$ are the constraint critical points.

If $\lambda = \pm 2$. The third equation tells us that $z = 0$. If $\lambda = 2$, $x = y$ and the last equation can't be satisfied. Hence $\lambda = -2$. Then, $x = -y, y = -x$. The first two equations are dealt with. The last equation then tells us that $x = 1, y = -1$ or $x = -1, y = 1$. Hence we have found two points $(1, -1, 0)$ and $(-1, 1, 0)$

Plugging these points into f, we see that $f(0, 0, \pm 1) = 1$ and $f(1, -1, 0) = 2, f(-1, 1, 0) = 2$. Now let's look at the function. As you see, when you go far on the surface, f becomes large. This means there must be minimum in the middle. Hence $(0, 0, \pm 1)$ must be the minimum points. However, there are no global maximum points. $(1, -1, 0)$ and $(-1, 1, 0)$ are not global max points.

(b). Find the minimum value of $f(x, y, z) = xyz$ under the constraint $x^2 + 2y^2 + z^2 = 1$.

Before calculation, you notice that the surface is closed and bounded. There must be max and min. Therefore, we can find the min by solving Lagrange multiplier equations.

On the surface, $\nabla g \neq \vec{0}$. Then, the only possibility is that $\nabla f = \lambda \nabla g, g = 1$ where $g(x, y, z) = x^2 + 2y^2 + z^2$

Then, you have four equations, $yz = \lambda 2x, xz = \lambda 4y, xy = \lambda 2z, x^2 + 2y^2 + z^2 = 1$. Multiplying the first equation with x, the second with y, the third with z, we have $\lambda 2x^2 = \lambda 4y^2 = \lambda 2z^2$, or $\lambda x^2 = \lambda 2y^2 = \lambda z^2$

If $\lambda = 0$, then $yz = 0, xz = 0, xy = 0$. There are 6 points. However, we see there's no need to solve these points as the value of f at these points must be $f = xyz = 0$.

If $\lambda \neq 0$, we have $x^2 = 2y^2 = z^2$. Hence $x^2 = 1/3, z^2 = 1/3, 2y^2 = 1/3$. Notice we care the value only. We see that $(xyz)^2 = 1/54$ and $xyz = -1/\sqrt{54}$ is the minimum. We see that this solution is not fake as $x = -1/\sqrt{3}, y = -1/\sqrt{6}, z = -1/\sqrt{3}$ satisfies the requirement. This is of course smaller than 0 as we get above. Hence the minimum value is $-1/\sqrt{54}$

(c). $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + z^2$. Find the largest and smallest value of f inside $x^2 + y^2 + z^2 \leq 16$.

Notice the domain is closed and bounded. The max and min must exist.

For interior extremum, we have $\nabla f = 0$ or $2(x-1) = 0, 2(y-2) = 0, 2z = 0$. We have one point $(1, 2, 0)$. We check that $1^2 + 2^2 + 0^2 = 5 < 16$.

This is in fact an interior extremum. You can use 2nd derivative test to find the local behavior but we need the global extremum here and we just regard $(1,2,0)$ as the candidates without doing 2nd derivative test.

For boundary extremum, we should have $g(x,y,z) = x^2 + y^2 + z^2 = 16$. On the constraint, $\nabla g \neq 0$. Hence we have $\nabla f = \lambda \nabla g, g = 16$

$2(x-1) = \lambda 2x$, $2(y-2) = \lambda 2y$, $2z = \lambda 2z$, $x^2 + y^2 + z^2 = 16$. Look at the third equation, if $\lambda = 0$, we get $(1,2,0)$ which of course is not on the boundary. Hence $\lambda \neq 0$ and $z = 0$

From the first two equations, $x = 1/(1-\lambda), y = 2/(1-\lambda)$. Hence $y = 2x$. Then, $x^2 + y^2 + 0^2 = 16$ tells you that $x = \pm 4/\sqrt{5}$. Hence, we have two other candidates $(4/\sqrt{5}, 8/\sqrt{5}, 0)$ and $(-4/\sqrt{5}, -8/\sqrt{5}, 0)$

Provided the existence of max/min, we just need to compute the values at these points. $f(1,2,0) = 0$, $0 < f(4/\sqrt{5}, 8/\sqrt{5}, 0) < f(-4/\sqrt{5}, -8/\sqrt{5}, 0)$. Hence, 0 is the global minimum and $f(-4/\sqrt{5}, -8/\sqrt{5}, 0)$ is the global max.

(d). Let $f(x,y) = x + y - xy$. Let D be the region bounded by $x = 0, y = 0, x + 2y = 4$. Find the maximum and minimum values of f on D.

Also the region is bounded and closed. There must be max/min.

For interior, $\nabla f = (1-y, 1-x) = 0$. We get $(1,1)$. This is one candidate(it's a saddle actually).

Consider the boundary. We have three pieces. For $x = 0, 0 \leq y \leq 2$. The function is reduced to $0 + y - 0 = y$. Hence on this piece, the smallest value is obtained at $(0,0)$ which is 0 and the largest is obtained at $(0,2)$ which is 2. For $0 \leq x \leq 4, y = 0$, the function is reduced to x. Again, the minimum is 0 at $(0,0)$ and the maximum is 4 at $(4,0)$; For the piece $x + 2y = 4, 0 \leq x \leq 4$, the function is reduced to $x + (4-x)/2 - x(4-x)/2$. You see that the minimum value should be at $x = 3/2, y = 5/4$ and the largest value is at $(4,0)$.

Anyway, combining all these together, we see that $(0,0)$ is the global minimum and $(4,0)$ is the global max.

You can use Lagrange multiplier for each piece which is more complicated in this problem.

3. (Double integral, iterated integral)

(a). Compute $\iint_D \sqrt{x^3+1}dA$ where $D = \{(x,y) : 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1\}$

Writing in iterated integral form:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy$$

That is not suitable for integration. We change the order and have

$$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}dydx = \int_0^1 x^2\sqrt{x^3+1}dx = \frac{2}{9}(2\sqrt{2}-1)$$

(b). Compute

$$\int_0^1 \int_{y^2}^1 y(3x^2+1)^{1/3}dxdy$$

Change the order:

$$\int_0^1 \int_0^{\sqrt{x}} y(3x^2+1)^{1/3}dydx = \int_0^1 \frac{1}{2}x(3x^2+1)^{1/3}dx = \frac{1}{16}(4^{4/3}-1)$$

(c). Evaluate $\iint_R \frac{y}{x^5+1}dA$ where R is the region bounded by $y=0, y=x^2, x=1$

$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1}dydx = \int_0^1 \frac{x^4/2}{x^5+1}dx = (\ln 2)/10$$

4. (Double Integrals in polar coordinates)

(a). Write the following curves and functions in polar coordinates:

- $x^2+(y-a)^2=a^2$
- $x^3+xy^2-y=0$
- $f(x,y)=2x^2+y^2-x$

The first: $r=2a\sin\theta$. The second is $r^2=\tan\theta$(think about why we can divide $\cos\theta$). The third is $F(r,\theta)=r^2+r^2\cos^2\theta-r\cos\theta$

(b). Find the volume of the region bounded by $z=x^2+y^2$ and $z=y$.

The integral is $\iint_D(y-x^2-y^2)dA$ where the boundary of D is determined by $x^2+y^2=y$

From here, you see that polar coordinates are more convenient: The boundary in polar is $r^2=r\sin\theta$ or $r=\sin\theta$. You see that from $\theta=0$ to $\theta=\pi$, you can trace the boundary once(You set $r=0$ and find $\theta=0,\pi$). The region can be written as $\{0\le\theta\pi, 0\le r\le\sin\theta\}$. $dA=rdrd\theta$. The integrand is $r\sin\theta-r^2$. Hence the integral is

$$\int_0^\pi \int_0^{\sin\theta}(r\sin\theta-r^2)rdrd\theta = \frac{1}{12}\int_0^\pi \sin^4\theta d\theta$$

To evaluate this integral, you recall double angle formula: $\sin^2\theta = (1-\cos 2\theta)/2$ and the integral is:

$$\frac{1}{48}\int_0^\pi (1 - 2\cos 2\theta + \cos^2(2\theta))d\theta = \frac{\pi}{32}$$

where the double angle formula is used again($\cos^2 2\theta = (1+\cos 4\theta)/2$).

(c). Let $R = \{(x,y) : (x-1)^2 + y^2 \le 1, x^2 + (y-1)^2 \le 1\}$. Compute the volume of under $f(x,y) = x$ and above R.

The region can be written in polar as $\{0 \le \theta \le \pi/4, 0 \le r \le 2\sin\theta\}$ together with $\{\pi/4 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}$. The volume $\iint_D xdA$ then can be written as

$$\int_0^{\pi/4}\int_0^{2\sin\theta} r\cos\theta(rdrd\theta) + \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r\cos\theta(rdrd\theta)$$

I think you can finish evaluating the integral.. For the first, use u-sub $u = \sin\theta$. The second piece, use double angle formula again.

(d). Compute $\int_0^1\int_0^{\sqrt{1-y^2}} \sin(x^2+y^2)dxdy$

In polar: the region is $\{0 \le \theta \le \pi/2, 0 \le r \le 1\}$. Hence the integral is

$$\int_0^{\pi/2}\int_0^1 \sin(r^2)rdrd\theta = \frac{\pi}{4}(1-\cos 1)$$

5. (Triple integrals)

(a). Evaluate $\iiint e^{x+y+z}dV$ over the region $\{(x,y,z) : 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \ln y\}$.

$$\begin{aligned}\int_0^1\int_0^x\int_0^{\ln y} e^{x+y+z}dzdydx &= \int_0^1\int_0^x (y-1)e^{x+y}dydx \\ &= \int_0^1 ((y-1)e^{x+y} - e^{x+y})|_0^x dx = \int_0^1 [(x-2)e^{2x} + 2e^x]dx = \dots\end{aligned}$$

(b). Evaluate $\iiint_D xydV$ where D is the region bounded by $y = x^2, x = y^2, z = 0, z = x+y$

$$\begin{aligned}\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xydzdydx &= \int_0^1\int_{x^2}^{\sqrt{x}} xy(x+y)dydx \\ &= \int_0^1 x^2\frac{1}{2}(x-x^4) + \frac{1}{3}x(x^{3/2} - x^6)dx = \dots\end{aligned}$$

(c). Compute the integral of $f(x, y, z) = x$ over the region $0 \le y \le 1, x, z \ge 0, x + z \le 2$

The region $\{(x, y, z) : 0 \le y \le 1, 0 \le x \le 2, 0 \le z \le 2 - x\}$.

$$\int_0^1 \int_0^2 \int_0^{2-x} x dz dx dy = \int_0^2 x(2-x)dx = \dots$$

(d). A cube has edge length 2 and density equals the square of the distance from one specific edge. Find the total mass and the center of mass.

$$M = \int_0^2 \int_0^2 \int_0^2 (y^2 + z^2) dx dy dz$$

$$x_c = \frac{1}{M} \int_0^2 \int_0^2 \int_0^2 x(y^2 + z^2) dx dy dz$$

$$y_c = \frac{1}{M} \int_0^2 \int_0^2 \int_0^2 y(y^2 + z^2) dx dy dz = z_c$$

6. (Triple integrals in cylindrical/spherical coordinates)

(a). (hw #14)Evaluate $\iiint x^2 dV$ over the interior of the cylinder $x^2+y^2 = 1$ between $z = 0$ and $z = 5$.

Use cylindrical coordinates. $x^2 = r^2 \sin^2 \theta$. Region is $0 \le z \le 5, 0 \le \theta \le 2\pi, 0 \le r \le 1$. $dV = r dr d\theta dz$. The integral is

$$\int_0^5 \int_0^{2\pi} \int_0^1 r^2 \sin^2 \theta r dr d\theta dz$$

(b). (hw #19) Evaluate $\iiint \sqrt{x^2 + y^2} dV$ over the interior of $x^2+y^2+z^2 = 4$

Us spherical coordinates. $dV = \rho^2 \sin \phi d\rho d\phi d\theta$

$$\int_0^2 \int_0^{\pi} \int_0^{2\pi} \sqrt{\rho^2 \sin^2 \phi} \rho^2 \sin \phi d\theta d\phi d\rho = \int_0^2 \int_0^{\pi} \int_0^{2\pi} \rho^3 \sin^2 \phi d\theta d\phi d\rho$$

(c). (hw #17) Evaluate $\iiint yz dV$ over the region in the first octant inside $x^2 + y^2 - 2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

Use cylindrical coordinates. $r^2 - 2r\cos\theta = 0$ or $r = 2\cos\theta$. The second equation is $r^2 + z^2 = 4$. The integral is

$$\int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{\sqrt{4-r^2}} (r\sin\theta)z[rdzdrd\theta]$$