Keys-Review Working sheet

1. (vectors. Dot/cross product)

   Suppose $A(0,0,1), B(2,1,3), C(-1,-1,0), D(-2,-4,5)$

   (a). Compute plane ABC

   (b). Distance from D to ABC

   (c). Angle between $\overrightarrow{AB}, \overrightarrow{AC}$

   (d). The volume of the parallelepiped that has edges AB, AC, AD

   Soln. (a). First of all, we have $\overrightarrow{AB} = (2,1,2)$ and $\overrightarrow{AC} = (-1,-1,-1)$. A normal vector could be found as $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$:

$$\vec{n} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

   The point on the plane could be picked as $A(0,0,1)$ and the plane could be computed as

$$\vec{n} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = 0$$

   or $1x + 0y - (z-1) = 0$ or $x - (z-1) = 0$

   (b). $\overrightarrow{AD} = (-2,-4,4)$ where A is on the plane. The distance could be computed as

$$d = \frac{|\vec{n} \cdot \overrightarrow{AD}|}{|\vec{n}|} = \frac{|-2+0-4|}{\sqrt{2}} = 3\sqrt{2}$$

   (c).

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{-2-1-2}{\sqrt{4+1+4}\sqrt{1+1+1}} = \frac{-5}{3\sqrt{3}}$$

   Hence $\theta = \arccos(-5/(3\sqrt{3})) = \pi - \arccos(5/(3\sqrt{3}))$

   (d).

$$V = |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}| = 6$$

2. (Parametrized curve; This is from the exam of last year I was TAing for.)

Consider

$$\vec{x}(t) = \begin{pmatrix} t\sin t + \cos t \\ t\cos t - \sin t \\ t^2 \end{pmatrix}, t \geq 0$$

Compute the unit tangent, unit normal and curvature. Also compute the arclength from $t = 1$ to $t = 2$

Soln. First of all, we can compute that

$$\vec{x}'(t) = \begin{pmatrix} t\cos t \\ -t\sin t \\ 2t \end{pmatrix} = t \begin{pmatrix} \cos t \\ -\sin t \\ 2 \end{pmatrix}$$

Then, we compute that $|\vec{x}'(t)| = t\sqrt{\cos^2 t + \sin^2 t + 4} = \sqrt{5}t$.

The unit tangent is

$$\vec{T}(t) = \frac{\vec{x}'(t)}{|\vec{x}'(t)|} = \frac{1}{\sqrt{5}} \begin{pmatrix} \cos t \\ -\sin t \\ 2 \end{pmatrix}$$

The curvature vector is

$$\vec{\kappa}(t) = \frac{d}{ds}\vec{T} = \frac{1}{|\vec{x}'(t)|}\vec{T}'(t) = \frac{1}{\sqrt{5}t}\frac{1}{\sqrt{5}} \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} = \frac{1}{5t} \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}$$

The curvature is

$$\kappa = |\vec{\kappa}| = \frac{1}{5t} | \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} | = \frac{1}{5t}$$

The unit normal could be computed as $\vec{N} = \vec{\kappa}/\kappa$. Or, we simply take the direction of $(-\sin t, -\cos t, 0)$, which is

$$\vec{N} = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}$$

Finally, the arc length is

$$L = \int_1^2 |\vec{x}'(t)| dt = \sqrt{5}\frac{1}{2}t^2|_1^2 = \frac{\sqrt{5}}{2}(4-1)$$

A final comment: if we are not asked about the normal, we can compute the curvature as $\kappa = \frac{1}{|\vec{x}'(t)|^3}\|\vec{x}'(t) \times \vec{x}''(t)\|$.

3. (Quadratic form)

$$Q(x, y) = x^2 + 6xy + 8y^2$$

Which kind of form is this?

Soln. Since $A = 1, B = 6, C = 8$ and we have $4AC - B^2 = 32 - 36 < 0$, it's indefinite.

Another way is to complete the square:

$$Q = (x^2+6xy+9y^2-9y^2)+8y^2 = (x+3y)^2-9y^2+8y^2 = (x+3y)^2-y^2 = u^2-v^2$$

4. (Polar coordinates)

(a). If $x < 0, y > 0$, find a formula for the polar angle θ. Use your formula to find r, θ for point $(-1, \sqrt{3})$

(b). Sketch the surface $x^2 + y^2 - z^2 = 0$

Soln. (a). Notice that $\tan\theta = y/x < 0$ and $\arctan(y/x) \in (-\pi/2, 0)$. To get the correct angle, we fix it as:

$$\theta = \arctan(\frac{y}{x}) + \pi$$

For $(-1, \sqrt{3})$, we have $\theta = \arctan(-\sqrt{3}) + \pi = 2\pi/3$. $r = \sqrt{1^2 + 3} = 2$

(b). We can solve the surface as $z = r$ and $z = -r$. You plot the graph in zx plane first and rotate about z-axis. You'll get a cone.(figure omitted.)

5. (This is also an old exam; the purpose is to test partial derivatives. You can choose to use linear approximation)

Compute the limit

$$\lim_{h \to 0} \frac{f(x-h, y+h) - f(x, y)}{h}$$

Soln. I choose to use linear approximation

$$\begin{aligned} f(x-h, y+h) &= f(x, y) + f_x(x, y)[x - h - x] + f_y[y + h - y] + O(h^2) \\ &= f(x, y) - hf_x(x, y) + hf_y(x, y) + O(h^2) \end{aligned}$$

Plugging in, we have

$$\lim_{h \to 0}(-f_x(x, y) + f_y(x, y) + O(h)) = -f_x(x, y) + f_y(x, y)$$

6. (linear approximation, tangent of the graph, tangent of the level set)

(a). Consider

$$f(x,y) = xe^{\sin y}$$

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- Compute the linear approximation at $(1,0)$ and compute the value $1.02 * e^{\sin(0.01)}$ approximately;
- compute the tangent plane of the graph at $(1,0)$;
- compute the tangent line of the level set that passes through $(1,0)$.

Soln. (first) The linear approximation is just the first several terms in the Taylor expansion:

$$f(x,y) \approx f(1,0) + f_x(1,0)(x-1) + f_y(1,0)y$$

We compute that $f(1,0) = 1e^0 = 1$, $f_x = e^{\sin y}$ and $f_x(1,0) = 1$. $f_y = xe^{\sin y}\cos y$ and $f_y(1,0) = 1$.

The linear approximation is thus

$$f(x,y) \approx 1 + (x-1) + y = x + y$$

We estimate that $1.02 * e^{\sin 0.01} = f(1.02, 0.01) \approx 1.02 + 0.01 = 1.03$ by linear approximation.

(second) The tangent plane of the graph is just the graph of the linear approximation of f. It's

$$z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)y = 1 + (x-1) + y = x + y$$

(third). The tangent line of the level set can be computed in two ways.

The first way is to use linear approximation. The equation for the level set is $f(x,y) - f(1,0) = 0$. Taking the linear approximation, we have

$$f_x(1,0)(x-1) + f_y(1,0)(y-0) = 0 \Rightarrow x - 1 + y = 0$$

The second way is to use the geometric meaning of gradient. The gradient is a normal vector of the level set and we see that $\nabla f(1,0) = (f_x(1,0), f_y(1,0)) = (1,1)$. The tangent line is

$$\nabla f \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0$$

or $(x-1) + y = 0$

(b). Consider

$$f(x, y, z) = xy^2z + (x + 4y + 2z)^3$$

.

- Compute the linear approximation at $(2, -1, 1)$;
- Compute the tangent hyperplane of the graph at $(2, -1, 1)$;
- compute the tangent plane of the level set that passes through $(2, -1, 1)$.

Soln. (first). The linear approximation is $f(x, y, z) \approx f(2, -1, 1) + f_x(2, -1, 1)(x - 2) + f_y(2, -1, 1)(y + 1) + f_z(2, -1, 1)(z - 1)$.

We compute that $f(2, -1, 1) = 2$ and $f_x(2, -1, 1) = 1, f_y(2, -1, 1) = -4, f_z(2, -1, 1) = 2$. The linear approximation is

$$f(x, y, z) \approx 2 + 1(x - 2) - 4(y + 1) + 2(z - 1)$$

(2nd) The tangent hyperplane of the graph is

$$\begin{aligned} w = f(2, -1, 1) + f_x(2, -1, 1)(x-2) + f_y(2, -1, 1)(y+1) + f_z(2, -1, 1)(z-1) \\ = 2 + 1(x - 2) - 4(y + 1) + 2(z - 1) \end{aligned}$$

(3rd). The tangent of the level set can be computed using two ways as well. The first way is to use linear approximation of the level set equation $f(x, y, z) - f(2, -1, 1) = 0$ and get

$$f_x(2, -1, 1)(x-2) + f_y(2, -1, 1)(y+1) + f_z(2, -1, 1)(z-1) = 0 \Rightarrow 1(x-2) - 4(y+1) + 2(z-1) = 0$$

The second way is to use the geometric meaning of gradient as well(gradient is a normal of level set).

$$\nabla f(2, -1, 1) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) = 0$$

or $1(x - 2) - 4(y + 1) + 2(z - 1) = 0$.

7. (Gradient; revised from homework)

Consider $f(x, y) = x^2 + y^2 + z$.

(a). Compute ∇f.

(b). Find the fastest increasing direction and fastest decreasing direction at $(0, 0, 1)$.

(c). Let's say the level set that passes through $(0, 0, 1)$ is C. Find a point on C where the normal of the level set is parallel with $(1, 1, 1)$.

Soln. (a). It's easy to see that

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$$

(b). Since the gradient points the fastest increasing direction, we see that

$$\hat{v} = \frac{\nabla f}{|\nabla f|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is the fastest increasing direction. The fastest decreasing direction is $-\hat{v} = (0, 0, -1)$.

(c). The level set is $f(x, y, z) = C$ and we can compute that $C = 0^2 + 0^2 + 1 = 1$. Thus the level set is $x^2 + y^2 + z = 1$.

Then, by the formula of ∇f, we hope $(2x, 2y, 1) \parallel (1, 1, 1)$. That means we should have $(2x, 2y, 1) = \lambda(1, 1, 1)$. Thus $x = 1/2, y = 1/2$. However, for the point, we need z to make $(1/2, 1/2, z)$. To find z, we use the level set equation $x^2 + y^2 + z = 1$ and find that $z = 1 - 1/4 - 1/4 = 1/2$

8. (Chain rule)

(a). Let $f(x,y) = \cos(xy) + y\cos(x^2)$. Suppose a bug is crawling along a curve $\vec{x}(t) = (t^2, t)$. The bug then feels the f as $f(x(t), y(t))$. Compute the changing rate that the bug feels: df/dt.

(b). Let $f(x,y) = \cos(xy) + y\cos(x^2)$ and $x = s+t, y = s/t$. $z = g(s,t) = f(x(s,t), y(s,t))$. Compute $\partial z/\partial s = g_s$ and $\partial z/\partial t = g_t$.

(c*). Consider the Cartesian point $(1,1)$. If you know $\partial f/\partial r = 1$ and $\partial f/\partial\theta = 2$ when regarding f as a function of r, θ, could you compute $\partial f/\partial x$ when you regard f as a function of x, y?

Soln. (a). First of all, we notice $x(t) = t^2, y(t) = t$. By the formula of chain rule, we have

$$\frac{df}{dt} = f_x x'(t) + f_y y'(t) = [-y\sin(xy) - 2xy\sin x^2]2t + [-x\sin(xy) + \cos(x^2)]1$$

Plugging in $x = t^2, y = t$, we have

$$\frac{df}{dt} = [-t\sin(t^3) - 2t^3\sin t^4]2t + [-t^2\sin(t^3) + \cos(t^4)]1$$

(b). By the formula, we have $g_s = f_x\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s}$ or

$$g_s = [-y\sin(xy) - 2xy\sin x^2]1 + [-x\sin(xy) + \cos(x^2)](1/t)$$

or

$$g_s = [-(s/t)\sin((s+t)s/t) - 2(s+t)s/t\sin(s+t)^2]1 + [-(s+t)\sin((s+t)s/t) + \cos((s+t)^2)](1/t)$$

Similarly, we have

$$g_t = [-y\sin(xy) - 2xy\sin x^2]1 + [-x\sin(xy) + \cos(x^2)](-s/t^2)$$

or

$$[-(s/t)\sin((s+t)s/t) - 2(s+t)s/t\sin(s+t)^2]1 + [-(s+t)\sin((s+t)s/t) + \cos((s+t)^2)](-s/t^2)$$

(c). Let's consider $f(x,y) = f(x(r,\theta), y(r,\theta))$, where $x = r\cos\theta, y = r\sin\theta$. We then have

$$f_r = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = \cos\theta f_x + \sin\theta f_y$$

Similarly, we have

$$f_\theta = f_x(-r\sin\theta) + f_y(r\cos\theta)$$

At the point, we have $r = \sqrt{2}, \theta = \pi/4$. Thus,

$$1 = f_r = \frac{\sqrt{2}}{2}f_x + \frac{\sqrt{2}}{2}f_y$$
$$2 = f_\theta = -f_x + f_y$$

Then, just solve the system of equations.

9. (Implicit function)

Compute $\partial z/\partial x$ at $(1, -1, 1)$ if $\ln(2x + yz) - xy - zx = 0$.

Soln. Let's re-derive the formula again:

$$F(x, y, z(x, y)) = C$$

Taking partial derivative on x on both sides(that means we are fixing y), we have

$$F_x + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

We compute that

$$F_z = \frac{1}{2x + yz} y - x = \frac{1}{2 - 1} * (-1) - 1 = -2 \neq 0$$

The function exists locally around $(1, -1, 1)$.

Then,

$$F_x = \frac{2}{2x + yz} - y - z = \frac{2}{1} + 1 - 1 = 2$$

The partial derivative is $-(2)/(-2) = 1$.

10. (Finding function from its partial derivatives; higher order derivatives)

Suppose your region is a ball. Inside this ball, you have

$$P(x,y) = \frac{2x}{x^2+1} + 2xy^3 + y\cos(xy)$$
$$Q(x,y) = 3x^2y^2 + x\cos(xy) + e^y$$

Is there a function f so that $f_x = P$ and $f_y = Q$? If yes, find all such functions and compute f_{xy}, f_{yy}.

Soln. We check if $P_y = Q_x$ holds or not.

$$P_y = 0 + 6xy^2 + \cos(xy) - xy\sin(xy)$$

and

$$Q_x = 6xy^2 + \cos(xy) - xy\sin(xy) + 0$$

They are equal. The domain doesn't have holes inside and there must be such functions.

$$f(x,y) = \int P dx = \ln(x^2+1) + x^2y^3 + \sin(xy) + g(y)$$

Then, $Q = f_y$ means

$$3x^2y^2 + x\cos(xy) + e^y = 3x^2y^2 + x\cos(xy) + g'(y)$$

or $g(y) = \int e^y dy = e^y + C$. In other words, we have

$$f(x,y) = \ln(x^2+1) + x^2y^3 + \sin(xy) + e^y + C$$