Exercises on Nov. 11th

1. Find the largest surface area of a rectangular box without top, provided that the diagonal is A

Assume the length, width, height are $x, y, z$ respectively. The surface area is $f(x, y, z) = xy + 2xz + 2yz$. The diagonal is $\sqrt{x^2 + y^2 + z^2} = A$ and hence $g(x, y, z) = x^2 + y^2 + z^2 = A^2$

Clearly, $\nabla g \neq 0$ and thus $f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, g = A^2$.

$y + 2z = \lambda 2x, x + 2z = \lambda 2y, 2x + 2y = \lambda 2z, x^2 + y^2 + z^2 = A^2$

From the first equation, we solve $\lambda = (y + 2z)/(2x)$ (clearly, they are all nonzero). Plug this into the second equation, we get $y^2 + 2yz = x^2 + 2xz$ or $y^2 - x^2 + 2yz - 2xz = 0$ or $(y - x)(y + x + 2z) = 0$. Hence, $y = x$. This way may be hard for some people. Then, you can solve $2z = 2\lambda x - y$ and plug into the second and you have $x = y$ as well.

Now, using the same trick, you have $yz - 2x^2 = 2xy - 2z^2$. Using the condition that $x = y$, you have $2z^2 + xz - 4x^2 = 0$. Using the quadratic formula, $z = (-x + \sqrt{x^2 + 32x^2})/4 = (\sqrt{33} - 1)x/4$.

The constraint tells us that $x^2 + x^2 + \frac{33-2\sqrt{33}}{16}x^2 = A^2$. Now, you can solve $x = y = 4A/\sqrt{66 - 2\sqrt{34}}$. Then, you can have $z$. The final answer can be obtained by plugging in.

*I didn’t notice the computation is so tricky for you...*

2. The volume bounded by $y^2 = 4x, 2x + y = 4, y = 0, z = y, 2z = y$.

The intersection between $y^2 = 4x$ and $2x + y = 4$ can be obtained by solving $y^2 = 2(4 - y)$. Then, $(1, 2)$ is the intersection. The integral is

$$
\int_0^2 \int_{y^2/4}^{2-y/2} (y - y/2) dx dy = \int_0^2 \frac{y}{2} (2 - y/2 - y^2/4) dy = 5/6
$$

3. $R$ is the region bounded by $x = y^2, y = 1, x + y - z = 0$ in first octant. Density is $\mu(x, y, z) = x$. Find the total mass and average of $\mu$.

The region is $0 \leq y \leq 1, 0 \leq x \leq y^2, 0 \leq z \leq x + y$. The total mass is

$$
M = \iiint \mu dV = \int_0^1 \int_0^{y^2} \int_0^{x+y} x dz dx dy = \int_0^1 \left( \frac{1}{3} y^6 + \frac{1}{2} y^5 \right) dy = \frac{1}{21} + \frac{1}{12} = 11/84
$$

The total volume is

$$
V = \int_0^1 \int_0^{y^2} \int_0^{x+y} dz dx dy = \int_0^1 (y^4/2 + y^3) dy = 1/10 + 1/4
$$

and the average is $(11/84)/(7/20)$

4. The region is $x^2 + y^2 \geq 1$, $x^2 + y^2 \leq 4$, $|z| \leq 1$, $x \geq 0$. The density is $\mu = x^2 + y^2 + z^2$. Find the total mass.

Use cylindrical: the region is $1 \leq r \leq 2$, $-\pi/2 \leq \theta \leq \pi/2$, $-1 \leq z \leq 1$. $\mu = r^2 + z^2$. The total mass is

$$
\int_1^2 \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} (r^2 + z^2)r \, dz \, d\theta \, dr = \pi \int_1^2 (2r^3 + 2r/3) \, dr = \pi(15/2 + 1)
$$

5. Find the center of mass of the region $x^2 + y^2 + z^2 \leq 4$ and above $xy$-plane with density $\mu = \sqrt{x^2 + y^2}$.

By symmetry, the center of mass must be on $z$ axis. Hence $x_c = 0, y_c = 0$.

Now, let’s compute

$$
z_c = \frac{\iiint z \mu \, dV}{\iiint \mu \, dV}
$$

Use spherical. The region is $0 \leq \rho \leq 2, 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi$. The density is $\mu = \rho \sin \phi$ and $z = \rho \cos \phi$ in spherical. (You can plug in $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta$). Hence the integral is

$$
z_c = \frac{\int_0^2 \int_{\pi/2}^{\pi/2} \int_0^{2\pi} (\rho \cos \phi)(\rho \sin \phi) \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho}{\int_0^2 \int_{\pi/2}^{\pi/2} \int_0^{2\pi} (\rho \sin \phi) \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho}
$$

Numerator is

$$
2\pi \int_0^2 \int_0^{\pi/2} \cos \phi \sin^2 \phi \rho^4 \, d\phi \, d\rho = \frac{64\pi}{5} \int_0^{\pi/2} \sin^2 \phi \cos \phi \, d\phi = \frac{64\pi}{15}
$$

Notice that $u = \sin \theta$

For the denominator,

$$
2\pi \int_0^2 \int_0^{\pi/2} \rho^3 \sin^2 \phi \, d\phi \, d\rho = 8\pi \int_0^{\pi/2} \sin^2 \phi \, d\phi = 2\pi^2
$$

where $\sin^2 \phi = (1 - \cos(2\phi))/2$. Taking the ratio, you get the answer.