Exercises on Nov. 11th

1. Find the largest surface area of a rectangular box without top, provided that the diagonal is A

   Assume the length, width, height are x, y, z respectively. The surface area is $f(x, y, z) = xy + 2xz + 2yz$. The diagonal is $\sqrt{x^2 + y^2 + z^2} = A$ and hence $g(x, y, z) = x^2 + y^2 + z^2 = A^2$

   Clearly, $\nabla g \neq 0$ and thus $f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, g = A^2$.

   $y + 2z = \lambda 2x, x + 2z = \lambda 2y, 2x + 2y = \lambda 2z, x^2 + y^2 + z^2 = A^2$

   From the first equation, we solve $\lambda = (y + 2z)/(2x)$(clearly, they are all nonzero). Plug this into the second equation, we get $y^2 + 2yz = x^2 + 2xz$ or $y^2 - x^2 + 2yz - 2xz = 0$ or $(y - x)(y + x + 2z) = 0$. Hence, $y = x$. *This way may be hard for some people. Then, you can solve $2z = 2\lambda x - y$ and plug into the second and you have $x = y$ as well.*

   Now, using the same trick, you have $yz - 2x^2 = 2xy - 2z^2$. Using the condition that $x = y$, you have $2z^2 + xz - 4x^2 = 0$. Using the quadratic formula, $z = (-x + \sqrt{x^2 + 32x^2})/4 = (\sqrt{33} - 1)x/4$.

   The constraint tells us that $x^2 + x^2 + \frac{34-2\sqrt{33}}{16}x^2 = A^2$. Now, you can solve $x = y = 4A/\sqrt{66 - 2\sqrt{34}}$. Then, you can have z. The final answer can be obtained by plugging in.

   I didn't notice the computation is so tricky for you...

2. The volume bounded by $y^2 = 4x, 2x + y = 4, y = 0, z = y, 2z = y$.

   The intersection between $y^2 = 4x$ and $2x + y = 4$ can be obtained by solving $y^2 = 2(4 - y)$. Then, $(1, 2)$ is the intersection. The integral is

$$\int_0^2 \int_{y^2/4}^{2-y/2} (y - y/2)dxdy = \int_0^2 \frac{y}{2}(2 - y/2 - y^2/4)dy = 5/6$$

3. R is the region bounded by $x = y^2, y = 1, x + y - z = 0$ in first octant. Density is $\mu(x, y, z) = x$. Find the total mass and average of μ.

   The region is $0 \leq y \leq 1, 0 \leq x \leq y^2, 0 \leq z \leq x + y$. The total mass is

$$M = \iiint \mu dV = \int_0^1 \int_0^{y^2} \int_0^{x+y} xdzdxdy = \int_0^1 (\frac{1}{3}y^6 + \frac{1}{2}y^5)dy = \frac{1}{21} + \frac{1}{12} = 11/84$$

   The total volume is

$$V = \int_0^1 \int_0^{y^2} \int_0^{x+y} dzdxdy = \int_0^1 (y^4/2 + y^3)dy = 1/10 + 1/4$$

   and the average is $(11/84)/(7/20)$

4. The region is $x^2 + y^2 \geq 1, x^2 + y^2 \leq 4, |z| \leq 1, x \geq 0$. The density is $\mu = x^2 + y^2 + z^2$. Find the total mass.

   Use cylindrical: the region is $1 \leq r \leq 2, -\pi/2 \leq \theta \leq \pi/2, -1 \leq z \leq 1$. $\mu = r^2 + z^2$. The total mass is

$$\int_1^2 \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} (r^2 + z^2) r dz d\theta dr = \pi \int_1^2 (2r^3 + 2r/3) dr = \pi(15/2 + 1)$$

5. Find the center of mass of the region $x^2 + y^2 + z^2 \leq 4$ and above xy plane with density $\mu = \sqrt{x^2 + y^2}$.

   By symmetry, the center of mass must be on z axis. Hence $x_c = 0, y_c = 0$. Now, let's compute

$$z_c = \iiint z\mu dV / \iiint \mu dV$$

   Use spherical. The region is $0 \leq \rho \leq 2, 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi$. The density is $\mu = \rho \sin\phi$ and $z = \rho\cos\phi$ in spherical. (You can plug in $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta$). Hence the integral is

$$z_c = \frac{\int_0^2 \int_0^{\pi/2} \int_0^{2\pi} (\rho\cos\phi)(\rho\sin\phi)\rho^2 \sin\phi d\theta d\phi d\rho}{\int_0^2 \int_0^{\pi/2} \int_0^{2\pi} (\rho\sin\phi)\rho^2 \sin\phi d\theta d\phi d\rho}$$

   Numerator is

$$2\pi \int_0^2 \int_0^{\pi/2} \cos\phi \sin^2\phi \rho^4 d\phi d\rho = \frac{64\pi}{5} \int_0^{\pi/2} \sin^2\phi \cos\phi d\phi = \frac{64\pi}{15}$$

   Notice that $u = \sin\theta$

   For the denominator,

$$2\pi \int_0^2 \int_0^{\pi/2} \rho^3 \sin^2\phi d\phi d\rho = 8\pi \int_0^{\pi/2} \sin^2\phi d\phi = 2\pi^2$$

   where $\sin^2\phi = (1 - \cos(2\phi))/2$. Taking the ratio, you get the answer.