Exercise on September 16

Given

$$\vec{x}(t) = \begin{pmatrix} t \\ \frac{4}{3}t^{3/2} \\ \frac{1}{2}t^2 \end{pmatrix}$$

Compute the arc length from $t = 0$ to $t = 1$.

Soln:

First of all, we have

$$|\vec{x}'(t)| = \sqrt{1 + 4t + t^2} = \sqrt{(t+2)^2 - 3}$$

where we have completed the square $t^2+4t+1 = t^2+4t+2^2-2^2+1 = (t+2)^2-3$.

The arc length is

$$L = \int_0^1 \sqrt{(t+2)^2 - 3}dt = \int_2^3 \sqrt{u^2 - 3}du$$

where u-sub $u = t + 2$ has been used.

Now, to evaluate $\int \sqrt{x^2 - a^2}dx$ we do trig sub $x = a \sec\theta$. In our case, we do $u = \sqrt{3}\sec\theta$ and $du = \sqrt{3}\sec\theta\tan\theta d\theta$. The arc length is therefore reduced to

$$\int_{\sqrt{3}\sec\theta=2}^{\sqrt{3}\sec\theta=3} \sqrt{3\sec^2\theta - 3}\sqrt{3}\sec\theta\tan\theta d\theta = 3\int_{\sqrt{3}\sec\theta=2}^{\sqrt{3}\sec\theta=3} \sec\theta\tan^2\theta d\theta$$

By the formula of integration by parts $\int udv = uv - \int vdu$:

$$I = \int \sec\theta\tan^2\theta d\theta = \int \tan\theta * d(\sec\theta) = \tan\theta\sec\theta - \int \sec\theta d(\tan\theta)$$

where $v = \sec\theta$ and $u = \tan\theta$. The right hand side is

$$\begin{aligned} \tan\theta\sec\theta - \int \sec^3\theta d\theta &= \tan\theta\sec\theta - \int \sec\theta(\tan^2\theta + 1)d\theta \\ &= \tan\theta\sec\theta - I - \int \sec\theta d\theta \quad (1) \end{aligned}$$

In other words, we have

$$2I = \tan\theta\sec\theta - \int \sec\theta d\theta = \tan\theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

Since $\sec\theta = u/\sqrt{3}$, we have $\tan\theta = \sqrt{u^2-3}/\sqrt{3}$. Therefore, we actually have

$$3\frac{1}{2}\Big[\tan\theta\sec\theta - \ln|\sec\theta + \tan\theta|\Big]|_{\sqrt{3}\sec\theta=2}^{\sqrt{3}\sec\theta=3} = \frac{3}{2}\Big[\frac{u\sqrt{u^2-3}}{3} - \ln|\frac{u}{\sqrt{3}} + \frac{\sqrt{u^2-3}}{\sqrt{3}}|\Big]|_{u=2}^{u=3}$$

What I get is $\frac{3}{2}\sqrt{6} - 1 - \frac{3}{2}\ln(1 + \sqrt{6}/3)$