Summary for part 3

The definitions of double and triple integrals: limits of the corresponding Riemann sums. Fubini says they are equal to iterated integrals

1 Double integrals

$\iint_D f(x,y)dA$. Several things:

- You should be able to change the order of integration
- Area element in polar coordinates $(r,\theta) : x = r\cos\theta, y = r\sin\theta$ is $dA = rdrd\theta$ while in Cartesian $dA = dxdy$
- The volume of a region

$$V = \iiint_R dxdydz = \iint_D height\ dA$$

2 Volume Integrals

$\iiint_R f dV$. Several things:

- Change order of integration if necessary
- Cartesian $dV = dxdydz$
- Cylindrical $(r,\theta,z) : x = r\cos\theta, y = r\sin\theta, z = z$. The volume element is $dV = rdrd\theta dz$
- Spherical $(\rho,\phi,\theta) : x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. The volume element is $dV = \rho^2\sin\phi d\rho d\phi d\theta$
- Total mass $\iiint_V \mu dV$; Center of mass; etc

Vector field–It's just a vector-valued function in R^2 or R^3.(You associate a vector for each point)

3 Line integrals

Have covered three types of line integrals(there are other types)
A. Line integral of a scalar function $\int_C f(x,y)ds$
B. Line integral of a vector field $\int_C \vec{F} \cdot d\vec{x}$
C. Flux integral $\int_C \vec{v} \cdot \vec{N} ds$ $\vec{N}$ is the outer unit normal vector.
For 2 **dimensions**:

$$d\vec{x} = \vec{T} ds = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\vec{N} ds = \begin{pmatrix} dy \\ -dx \end{pmatrix}$$

$$ds = |d\vec{x}| = \sqrt{dx^2 + dy^2}$$

The second relation is true only if the angle is 90 clockwisely from $\vec{T}$ to $\vec{N}$, which is usually the case when we compute flux.

For 3 **dimensions**, $\vec{x} = (x, y, z)$, you'll have $d\vec{x} = (dx, dy, dz)$. $\vec{F} = (P, Q, R)$, you'll have $\vec{F} \cdot d\vec{x} = Pdx + Qdy + Rdz$. The second integral then becomes $\int_C (Pdx + Qdy + Rdz)$.

3.1 How to compute them generally?

Use parametrization
If C is given by $\vec{x} = \vec{x}(t)$, then

$$d\vec{x} = \vec{x}'(t)dt = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt$$

$$ds = |d\vec{x}| = |\vec{x}'(t)|dt = \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$\vec{N} ds = \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix} dt$$

Then, $\int_C f(x,y)ds = \int_a^b f(x(t), y(t))|\vec{x}'(t)|dt$ and the second one is $\int_a^b \vec{F} \cdot \vec{x}'(t)dt$

Use the meaning the integrals $\int ds = Length$, $\iint dA = Area$ etc
Application: The average of f on C is $\int_C fds / \int_C ds = \int_C fds / Length(C)$

3.2 Fundamental theorem (line integral version)

$$\int_C \nabla f \cdot d\vec{x} = \int_C f_x dx + f_y dy + f_z dz = \int_C df = f(B) - f(A)$$

This tells us that

$$\oint_C \nabla f \cdot d\vec{x} = 0$$

when f is a single-valued smooth function(for example $f = \theta = \arctan(y/x)$ is not OK for curve around origin, as θ is not single-valued smooth function).

Here the circle means the integral is on a closed curve.

3.3 Conservative vector field

If the circulation satisfies

$$\oint_C \vec{F} \cdot d\vec{x} = 0$$

for any **closed** curve C, $\vec{F}$ is called a conservative field and $\vec{F} = \nabla f$ for some scalar function f(called **potential**).

If $\vec{F}$ is not conservative, then you must use $Green's$ theorem(2d) or $Stokes$ Theorem(3d version) to find the circulation.

Criteria for conservative fields:

- (Clairaut) For $\vec{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$, $Q_x - P_y = 0$ is required
- For 3D vector $\vec{F}$, $\nabla \times \vec{F} = 0$ is needed
- Sometimes, you can find f so that $\vec{F} = \nabla f$

Comments:

$$curl(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{e}_x + (P_z - R_x)\vec{e}_y + (Q_x - P_y)\vec{e}_z$$

Therfore , the second condition is also $R_y = Q_z, P_z = R_x, Q_x = P_y$

3.4 Green's Theorem

This is the theorem that transforms the line integrals on **closed** curve to a double integral over the region inside.

We consider $\vec{v} = \begin{pmatrix} P \\ Q \end{pmatrix}$

- (Curl form) This is about **counterclock circulation**:

$$\oint_C \vec{v} \cdot d\vec{x} = \oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA = \iint_R curl(\vec{v})_z dA$$

Notice that

$$curl(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\vec{e}_z$$

- (Divergence form) This is about **outer flux**:

$$\oint_C \vec{v} \cdot \vec{N} ds = \iint_R (P_x + Q_y)dA = \iint_R div(\vec{v})dA$$

Here $div(\vec{v}) = \nabla \cdot \vec{v}$, the dot product between the operator ∇ and $\vec{v}$.

Notice the left hand side is $\oint_C \vec{v} \cdot \begin{pmatrix} dy \\ -dx \end{pmatrix} = \oint_C Pdy - Qdx$, from where you can see the two versions are equivalent.

$\nabla \cdot \vec{v}$ is the source or sink of the vector field which balances the flux. $\nabla \cdot \vec{v} > 0$, field is expanding while $\nabla \cdot \vec{v} < 0$ indicates compressing field.

4 Surface Integrals

There are two types:

$$\iint_S f dA$$

$$flux: \iint_S \vec{v} \cdot \vec{N} dA$$

dA is called the area element. $\vec{N}$ is the unit outer normal. $\vec{N}dA = d\vec{S}$ is the directed area element.

4.1 How to compute?

To use parametrization(the surface patch)

$$\vec{x} = \vec{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Then, we have:

$$\vec{N} dA = \vec{x}_u \times \vec{x}_v dudv$$
$$\vec{N} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$$
$$dA = \|\vec{x}_u \times \vec{x}_v\| dudv$$

Plugging these back, you get a double integral.

Example: Area element of polar coordinates in xy plane. The position vector can be parametrized as

$$\vec{x}(r, \theta) = r\cos\theta\hat{x} + r\sin\theta\hat{y} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}$$

Then, $dA = \|\vec{x}_r \times \vec{x}_\theta\| drd\theta = rdrd\theta$. This is the same as we argued in last Chapter.

Generally, dA is not $rdrd\theta$ in cylindrical coordinates for curved surface. Above is true only for straight planes. For those surfaces, you must use $\|\vec{x}_r \times \vec{x}_\theta\| drd\theta$ to get dA

Example: Flux of curl of $\vec{F}$: This is the key component in Stokes theorem. Basically, you want to compute

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{N} dA$$

Let $\omega = (1, 2, 3)$ and $\vec{F} = \omega \times \vec{x} = (2z - 3y, 3x - z, y - 2x)$. Let S be the upper hemisphere with radius 2. Compute the flux of curl of $\vec{F}$ on S.

Soln. $\nabla \times \vec{F} = (2, 4, 6)$ using the formula.(This can be confirmed if you know the advanced identity $\nabla \times (\omega \times x) = (\nabla \cdot \vec{x})\omega - (\omega \cdot \nabla)\vec{x} = 2\omega$–This of course is not expected from you. You can just use the formula to compute this)

Then parametrize the surface $\vec{x}(\phi, \theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$ for $0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi$. $\vec{N} dA = \vec{x}_\phi \times \vec{x}_\theta d\phi d\theta$. Plugging this in and computing, the answer will be $4\pi * 6$

4.2 Divergence theorem(Gauss theorem)

This is about the flux on a **closed** surface:

$$\oiint_S \vec{v} \cdot \vec{N} dA = \iiint_R div(\vec{v}) dV = \iiint_R \nabla \cdot \vec{v} dV$$

This has the same explanation as the divergence form of Green's theorem.

4.3 Stokes Theorem

This is about the circulation on a **closed** curve in $3D$ space. It's Green's theorem in $3D$ space.

$$\oint_C \vec{v} \cdot d\vec{x} = \iint_S curl(\vec{v}) \cdot \vec{N} dA$$

Here S can be any surface that has the boundary C and $curl(\vec{v}) = \nabla \times \vec{v}$

The curl is just the cross product between ∇ and $\vec{v}$