Math234 Summary

The exam could also test other topics, so you can't only rely on this as guidance for review.

1 Basics of functions of several variables

- Find the domain of a function.

  Example: $\sqrt{4-2x^2-4y^2}$ has domain $x^2+2y^2 \leq 2$(This is Not square. You can't write as $a \leq x \leq b, c \leq y \leq d$)

- **Quadratic forms**

  Quadratic form $Q(x,y) = Ax^2+Bxy+Cy^2$ (no $Dx+Ex+F$, which is different from quadratic functions). Classify them into definite, indefinite and semi-definite. Indefinite→saddle, definite→bowl. Semidefinite is like curved paper(think about x^2).

  Method 1:

  Competing the square. Example: $\frac{1}{2}x^2+xy+y^2 = \frac{1}{2}(x^2+2xy)+y^2 = \frac{1}{2}(x^2+2xy+y^2)-\frac{1}{2}y^2+y^2 = \frac{1}{2}(x+y)^2+\frac{1}{2}y^2$ so positive definite.

  Method 2: Use B^2-4AC. If >0, indefinite. If $=0$, semidefinite. If <0, definite. As for positive or negative, look at the coefficients of square terms.

2 Derivatives

2.1 Continuity

(This is not emphasized, so I'm thinking maybe it won't be tested).

If a function can be made continuous at origin(If the limit exists, then it can be made continous). First of all, check the limits on some typical directions (x-axis, y-axis, y=x, etc). If they don't agree, then the limit doesn't exist. Otherwise, let $x = r\cos\theta, y = r\sin\theta$ and $r \to 0$ seeing if it has a limit.

2.2 Partial Derivatives, including second derivatives

This part is important. You should be able to do them correctly.

Examples: Find all $f_x, f_y, f_{xx}, f_{xy} = f_{yx}, f_{yy}$: $x\tan(xy)+x\cos(y)+y^3$, $x\ln(x^2y)$, $\theta = \arctan(y/x)$, $\sqrt{x^2+y^2+z^2}$, e^{x+y^2}.

Another type is the chain rule: Find f_x and f_y
$z = f(xy^2)$, $z = f(y)g(x)$, $z = f(xy)g(y)$.
Soln to $z = f(xy)g(y)$: $f_x = yf'(xy)g(y), f_y = xf'(xy)g(y) + f(xy)g'(y)$

2.3 Gradient and Chain rule

Gradient is a vector. For $f(x, y)$,

$$\nabla f = (f_x, f_y)$$

For $f(x, y, z)$,

$$\nabla f = (f_x, f_y, f_z)$$

Gradient points the fastest increasing direction and perpendicular with level set. The magnitude is the changing rate, therefore the gradient will be bigger at the places where level curves are denser.

Examples: $Sec10.\#2, \#4, \#7, \#13$.

Example: If $\phi = -\ln(r)$ where $r = \sqrt{x^2 + y^2}$, find $\nabla\phi$.

Chain rule is very important.

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) = \nabla f \cdot \vec{x}'(t)$$

and

$$\frac{\partial f}{\partial u}(x(u, v), y(u, v)) = f_x\frac{\partial x}{\partial u} + f_y\frac{\partial y}{\partial u}$$

Example: Define $P = f(x, y)$. $x = st, y = s/t$. Find $\frac{\partial P}{\partial t}$.

Example: $f(x, y)$ is a function of x, y. Let $g = f(uv, u/v)$. Compute g_{uu}, g_{uv}, g_{vv}

Application 1:

Polar coordinate $x = r\cos\theta, y = r\sin\theta$. Example: If you know $\partial f/\partial r = 3, \partial f/\partial\theta = 6$ at Cartesian point $(4, 3)$. Find ∇f.

Also, if $Q = f(r, \theta)$ is a function of polar coordinates. Find $\frac{\partial Q}{\partial x}$

Application 2: Rotated coordinate system. $x = X\cos\alpha - Y\sin\alpha$ and $y = X\sin\alpha + Y\cos\alpha$. (Read 11.2)

2.4 Taylor Theorem, Linear Approximation and Tangent planes/lines

Taylor Theorem says:

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$+\frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + errors$$

Linear approximation just keeps the linear terms only:

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The tangent plane of the graph is the function for this linear approximation:

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The tangent line of the level curve of this function is:

$$0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = \nabla f \cdot (x-x_0, y-y_0)$$

Or you can write $(x,y) = (x_0,y_0) + s(-f_y, f_x)$. Because, gradient is the normal.

The tangent plane of the **level set** of $f(x,y,z) = c$ is:

$$\nabla f \cdot (x-x_0, y-y_0, z-z_0) =$$
$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$$

This is true since gradient is **normal** vector.

2.5 Implicit differentiation

For $F(x,y) = 0$, if $F_y \neq 0$, then you can solve y in terms of x and you have:

$$\frac{\partial}{\partial x}F(x,y(x)) = 0$$

giving $y'(x) = -\frac{F_x}{F_y}$. Similar thing is true when solving x in terms of y.

Example: $x^2 + 4y^2 + 16z^2 - 64 = 0$

At which points you can't solve z in terms of x,y? Suppose you are not at these points, find $\frac{\partial z}{\partial x}$ and $\frac{\partial^2 z}{\partial x^2}$.

Soln for the first part: $\partial F/\partial z = 32z$. So if $32z = 0$ or $x^2+4y^2-64 = 0$, we can't solve z in terms of x,y.

Example: $PV = nRT$. Find $\frac{\partial P}{\partial V}$, $\partial V/\partial T$ and $\partial T/\partial P$.

Soln for the first: Regard P as a function of V and T: $P_V V + P = 0$, so $P_V = -P/V$. This is the same if you solve $P = \frac{nRT}{V}$ and get $P_V = -\frac{nRT}{V^2} = -\frac{P}{V}$.

2.6 Finding functions using derivatives

Can we find f so that $f_x = P, f_y = Q$?
You just check if $P_y = Q_x$. If yes, integrate back.
Example: $P = -y/(x^2 + y^2), Q = x/(x^2 + y^2)$.
Example: $P = x/(x^2 + y^2), Q = y/(x^2 + y^2)$

3 Maxima and Minima

3.1 The concept, theory of local minimum/maximum and global maxmum/minimum

You also should know the theory that predicts the existence. Also, the extrema may come from boundary (example: $f(x, y) = x$).

3.2 Critical points and classifying them

We care critical points because they are candidates for global maxima and minima. We want to classify them into local min, local max and saddle.

If it's a function of two variables, $z = f(x, y)$. Use the second derivative test. Look at the Taylor theorem, we should focus on:

$$Q = \frac{1}{2} f_{xx}(x_0, y_0)\Delta x^2 + f_{xy}(x_0, y_0)\Delta x \Delta y + \frac{1}{2} f_{yy}(x_0, y_0)\Delta y^2$$

since the linear terms vanish at critical points.

Positive Definite: Opens up, so local min.

Negative definite: Opens down, so local max.

Indefinite: Can be both positive and negative, so saddle.

Semidefinite: Test fails. We should use other methods to figure out.

There's a method called Hessian test. Basically, the determinant of Hessian is $f_{xx}f_{yy} - f_{xy}^2$. This is exactly $-(B^2 - 4AC)$. If this is positive, , we have $B^2 - 4AC < 0$, we get definiteness. Similar conclusions for other types.

If the second derivative test fails or it's a function of more than 2 variables, you can fix x or fix y, or plot the graph and figure out how function changes.

3.3 Lagrange Multipliers

We want to find the maximum or minimum of f restricted on the level set of $g = c$. The conditions to be satisfied are:

$$\nabla f = \lambda \nabla g$$
$$g = c$$

or

$$\nabla g = 0$$
$$g = c$$

Example: Find the closest point on $4x^2 + 2y^2 + z^2 = 16$ to the origin.

You can solve this by setting $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 4x^2 + 2y^2 + z^2$

3.4 Application:Linear Regression

Read Section 7 in Chapter 5.