Discussion Chap 7(by Lei Li)

(Common mistake in hw: 2 minutes) In hw 10, #11, The pyramid with vertices $(-1,-1,0), (-1,1,0), (1,-1,0), (1,1,0)$ and $(0,0,2)$. Many people set integral as $\int_0^2 \int_{-1}^1 \int_{-1}^1$. This is wrong since it's rectangular box!

(5 minutes)How to set up one integral for the volume of a region by double integral? The formula:

$$V = \int_D height\ dA$$

Quiz last week: The volume of the region bounded by $y^2 = 4 - x$ and $y = 2z$ in first octant.

The height function at point (x, y) is determined by the second equation. Then,

$$height = y/2 - 0 = y/2$$

The region is determined by $y^2 = 4 - x$, namely $0 \le x \le 4, 0 \le y \le \sqrt{4-x}$. The integral is

$$\int_0^4 \int_0^{\sqrt{4-x}} \frac{y}{2} dydx$$

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A brief summary of what we have covered in vector calculus.

1. **Vector field**–It's just a vector-valued function in R^2 or R^3.(You associate a vector for each point)

2. **Line integrals** Have covered three types of line integrals(there are other types)

   A. Line integral of a scalar function $\int_C f(x, y) ds$

   B. Line integral of a vector field $\int_C \vec{F} \cdot d\vec{x}$

   C. Flux integral $\int_C \vec{v} \cdot \vec{N} ds$ $\vec{N}$ is the outer unit normal vector.

- What do they mean? Given the curve, you divide it into many pieces. For each piece, you compute the arclength ds or the displacement $d\vec{x}$. Then, find a value of the function or the vector field on this piece. Compute the integrand and then add together. As mesh tends to 0, if you have a limit, that limit is the corresponding line integral.

- How to compute?

  –Use parametrization. If C is given by $\vec{x} = \vec{x}(t)$, then $d\vec{x} = \vec{x}'(t)dt$ and $ds = |d\vec{x}| = |\vec{x}'(t)|dt$.

  Then, $\int_C f(x,y)ds = \int_a^b f(x(t), y(t))|\vec{x}'(t)|dt$ and the second one is $\int_a^b \vec{F} \cdot \vec{x}'(t)dt$

  –Use the meaning the integrals. $\int ds = Length$;

  $\vec{x} = (x, y, z)$, you'll have $d\vec{x} = (dx, dy, dz)$. $\vec{F} = (P, Q, R)$, you'll have $\vec{F} \cdot d\vec{x} = Pdx + Qdy + Rdz$. The second integral then becomes $\int_C (Pdx + Qdy + Rdz)$.

  Example: $A(1,2)$ and $B(2,3)$. C is the line segment connecting them. Let $\vec{F} = (x^2, x)$. Compute

  $$\int_C ds = Length\ of\ AB = \sqrt{1+1}$$

  $$\int_C \vec{F} \cdot d\vec{x} = \int_C (x^2 dx + x dy)$$

  Use parametrization $\vec{x} = (1,2) + t(1,1)$ to compute the second integral. Notice this is not equal to $(x^3/3 + xy)|_A^B$. The reason is that when you are integrating on the line, your x and y are changing at the same time.

  –Application: The average of f on C is $\int_C f ds / \int_C ds = \int_C f ds / Length(C)$

- **Fundamental theorem (line integral version)**

  $$\int_C \nabla f \cdot d\vec{x} = \int_C f_x dx + f_y dy + f_z dz = \int_C df = f(B) - f(A)$$

- **Conservative vector field** If for any **closed** curve C, you have:

  $$\oint_C \vec{F} \cdot d\vec{x} = 0$$

  then $\vec{F}$ is called a conservative field and $\vec{F} = \nabla f$ for some scalar function f

  Here the circle means the integral is on a closed curve.

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More examples and homework problems