Math234 Tangent planes and tangent lines

You should compare the similarities and understand them.

1. **Function of one variable**

   For $y = f(x)$, the tangent line is easy:

   $$y - f(x_0) = f'(x_0)(x - x_0)$$

   Actually, this comes from the linear approximation: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. The normal vector of this line is $(f'(x_0), -1)$.

   For vector function $\vec{x}(t)$, the tangent line is:

   $$\vec{r}(s) = \vec{x}(t_0) + s\vec{x}'(t_0)$$

2. **Function of two variables**

   For function $z = f(x, y)$.

   We can talk about the tangent plane of the graph, the normal line of the tangent plane(or the graph), the tangent line of the level curve, the normal line of the level curve.

   Tangent plane and the normal line of the graph are in xyz space while the things related to level curve are in xy plane.

   - **Tangent plane and normal line of graph**

     Tangent plane is:

     $$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

     This comes from the linear approximation of the function. $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

     The normal vector of this tangent plane is $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$. the **normal line** of the tangent plane:

     $$(x, y, z) = (x_0, y_0, f(x_0, y_0)) + s * (f_x(x_0, y_0), f_y(x_0, y_0), -1)$$

- **Tangent line and normal line of the level curve**

  We know the linear approximation is $f(x,y) \approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$. When we move on level curve $f(x,y) = f(x_0,y_0)$ and we get the tangent line of the level curve:

  $$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$$
  $$or\ \nabla f(x_0,y_0)\cdot(x-x_0,y-y_0)=0$$
  $$or\ (x,y)=(x_0,y_0)+s(-f_y(x_0,y_0),f_x(x_0,y_0))$$

  Geometrically, this makes perfect sense. **The gradient** is a **normal vector** of the curve. Therefore, the dot product between gradient and the displacement on the tangent line is 0.

  Alternatively, one can find the direction of the tangent line: $(-f_y, f_x)$ for example, and then you get the third equation.

  Because the gradient is the normal direction of the level curve, the **normal line** of the level curve is therefore:

  $$(x,y)=(x_0,y_0)+s\nabla f(x_0,y_0)$$

3. **Function of three variables** $f(x,y,z)$

   We can also talk about the tangent hyperplane and normal line of the graph. However, these are 4 dimensional structure and we ignore them. (You can guess that the hyperplane comes from the linear approximation as well $f(x,y,z)-f(x_0,y_0,z_0) \approx f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)$).

   Anyway, I only talk about the tangent plane of the level set and normal line of level set:

   The tangent plane of the level set is:

   $$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$
   $$or\ \nabla f(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0$$

   The first is obtained from the linear approximation if you move on level set. The second one is equivalent to the first one but it can also be understood if you know **gradient is the normal vector of level set**.

   The normal line of level set is

   $$(x,y,z)=(x_0,y_0,z_0)+s\nabla f(x_0,y_0,z_0)$$