Math234 Homework 5

If you find any mistakes, please contact Lei Li.

Chapter 3

- 8. The convention in the note package is: $0 \le \theta < 2\pi$ for polar angle.

  (a). The range of arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

  For $\theta = \arctan(\frac{y}{x})$, the domain doesn't contain $x = 0$ and we have $\tan(\theta) = \frac{y}{x}$. Also

$$\begin{cases} \theta \in (-\frac{\pi}{2}, 0) & x > 0, y < 0 \text{ } or \text{ } x < 0, y > 0 \\ \theta \in [0, \frac{\pi}{2}) & x > 0, y \ge 0 \text{ } or \text{ } x < 0, y \le 0 \end{cases}$$

  This tells us that $\arctan(\frac{y}{x})$ gives the polar angle when $x > 0, y \ge 0$. This region is the first quadrant and positive x-axis.

  (b). For $\theta = \pi + \arctan(\frac{y}{x})$, the domain doesn't contain $x = 0$ and we have

$$\tan(\theta) = \tan(\pi + \arctan(\frac{y}{x})) = \tan(\arctan(\frac{y}{x})) = \frac{y}{x}$$

  and

$$\begin{cases} \theta \in (\frac{\pi}{2}, \pi) & x > 0, y < 0 \text{ } or \text{ } x < 0, y > 0 \\ \theta \in [\pi, \frac{3\pi}{2}) & x > 0, y \ge 0 \text{ } or \text{ } x < 0, y \le 0 \end{cases}$$

  This gives the correct polar angle when $x < 0$(y arbitrary). This is the left half plane(second and third quadrants and the negative x-axis).

  (c). Similarly, the domain doesn't contain $x = 0$ and we have:

$$\tan(\theta) = \tan(2\pi + \arctan(\frac{y}{x})) = \tan(\arctan(\frac{y}{x})) = \frac{y}{x}$$

  and

$$\begin{cases} \theta \in (\frac{3\pi}{2}, 2\pi) & x > 0, y < 0 \text{ } or \text{ } x < 0, y > 0 \\ \theta \in [2\pi, \frac{5\pi}{2}) & x > 0, y \ge 0 \text{ } or \text{ } x < 0, y \le 0 \end{cases}$$

This function gives the correct polar angle for $x > 0, y < 0$ which is the fourth quadrant.

(d). The domain doesn't contain $y = 0$. We have

$$\cot(\theta) = \cot(\frac{\pi}{2} - \arctan(\frac{x}{y})) = \tan(\arctan(\frac{x}{y})) = \frac{x}{y}$$

$$\begin{cases} \theta \in (0, \frac{\pi}{2}] & x \leq 0, y < 0 \ or \ x \geq 0, y > 0 \\ \theta \in (\frac{\pi}{2}, \pi) & x > 0, y < 0 \ or \ x < 0, y > 0 \end{cases}$$

You can see that this works for $y > 0$(x arbitrary.) This is the upper half plane (First and second quadrants together with positive y axis.)

(e). The domain contains everything except $(0, 0)$. The range of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}}$$

This is exactly the sin value of the polar angle. Then, by the range of arcsin, this expression works for all $\{(x, y)|x \geq 0, y \geq 0\} \setminus \{(0, 0)\}$ which is the the first quadrant and the positive x,y axes.

- 9. $c = 0$ means the depth is 0. Therefore, $d^{-1}(0)$ is the boundary of the lake. $d^{-1}(24)$ is a closed curve in the picture. The depth has only nonnegative values, so $d^{-1}(-24)$ is the empty set. Since the maximal depth is 25.3, $d^{-1}(400)$ is also the empty set.

- 10. The graph of $f(x, y) = g(\sqrt{x^2 + y^2})$ is the surface obtained by rotating the part of graph of $y = g(x)$ that is on $x \geq 0$ about z-axis.

  The level sets will be a family of circles.

  These facts can be obtained if you notice $f(x, y) = g(r)$. That means f is a radially symmetric function. The value will be the same as long as the distance of (x, y) from the origin is the same.

  If $g(x) = x$, it's a straight line. $f(x, y) = \sqrt{x^2 + y^2}$ is the cone rotated from $y = x, x \geq 0$.

- 11. (a). If the square roots want to make sense in the scope of real numbers, we must have

$$9 - x^2 \geq 0$$
$$y^2 - 4 \geq 0$$

yielding the largest domain:

$$\{(x, y)| - 3 \leq x \leq 3, y \leq -2 \ or \ y \geq 2\}$$

This is two stripes.

(c). $\{(x, y)|x \geq 0, y \geq 0\}$. This is the right upper corner of the plane.

(e). For those to make sense:

$$xyz \geq 0$$
$$\sqrt{xyz} \neq 0$$

yielding the domain: $\{(x, y, z)|xyz > 0\}$

(f). We must have:

$$x^2 + 4y^2 \leq 16$$

The domain is $\{(x, y)|x^2+4y^2 \leq 16\}$ which is the boundary and interior of an ellipse.

- 12. The first one is paraboloid and the second one is the cone.

The paraboloid is $z = r^2$. You can see when r is small $z'(r) = 2r < 1$, the value of the function grows slowly and the level sets will be sparse in this region. When r is big $z'(r) = 2r > 1$, the function grows more quickly and the level sets will be dense. This is exactly what the first figure describes.

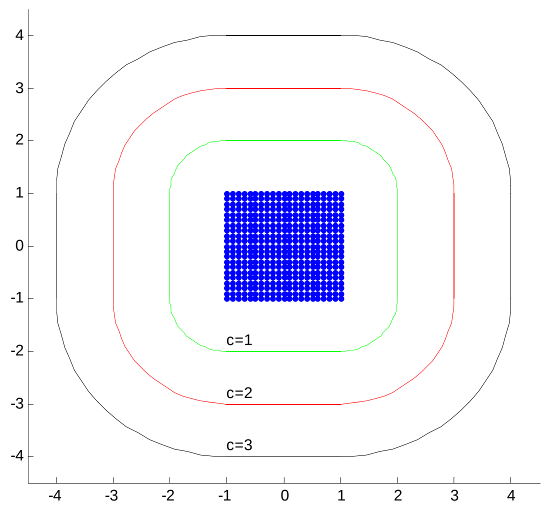
The cone is $z = r$ which grows linearly with the radius. The level sets will have equal distances between them. The second figure satisfies this.

- 13. (a). $(0, 1/2)$ is inside Q and therefore $(0, 1/2)$ is nearest to $(0, 1/2)$. By picture(not shown here), $(0, 1)$ is nearest to $(0, 2)$ and $(1, 1)$ is nearest to $(3, 4)$.

(b). $f(0,1/2) = 0$, $f(0,2) = \sqrt{(0-0)^2+(1-2)^2} = 1$, $f(3,4) = \sqrt{(3-1)^2+(4-1)^2} = \sqrt{13}$

(c). The zero set of f is Q since for every point inside Q, the closest point is itself and the distance is 0 and for any point that doesn't belong to Q, there is a minimum positive distance from the point to any point in Q(This is a nice property for closed set).

(d). The level set for -1 is empty set and therefore it's not on the picture. For the other three, see the picture below.

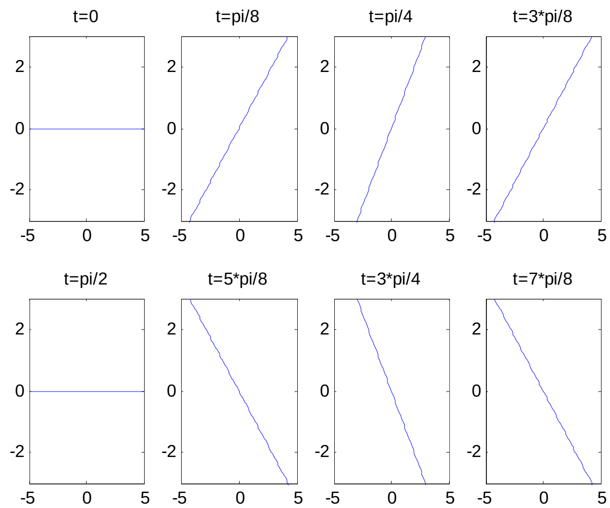


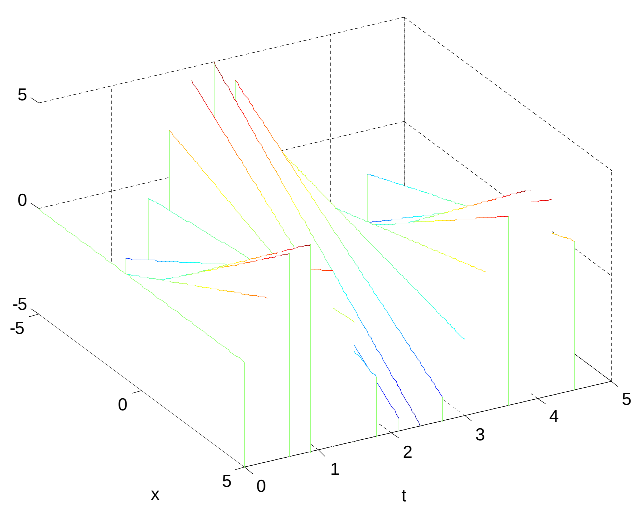
(e). By the picture above, it's clear that we should divide the whole plane into 9 regions. The corresponding distances are:

$$d = \begin{cases} 0 & (x,y) \in Q \\ x-1 & x>1, |y|<1 \\ y-1 & y>1, |x|<1 \\ -1-x & x<-1, |y|<1 \\ -1-y & |x|<1, y<-1 \\ \sqrt{(x-1)^2+(y-1)^2} & x \geq 1, y \geq 1 \\ \sqrt{(x+1)^2+(y-1)^2} & x \leq -1, y \geq 1 \\ \sqrt{(x+1)^2+(y+1)^2} & x \leq -1, y \leq -1 \\ \sqrt{(x-1)^2+(y+1)^2} & x \geq 1, y \leq -1 \end{cases}$$

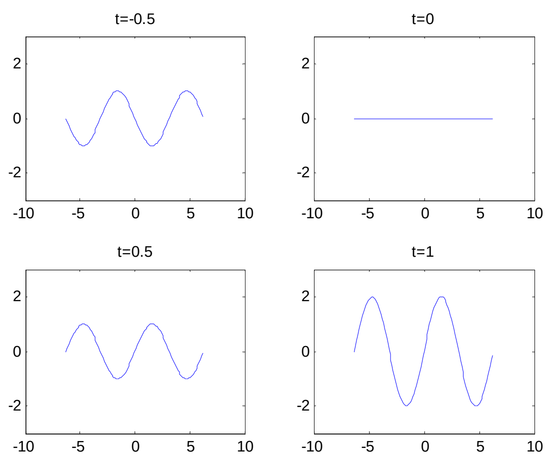
- 14.

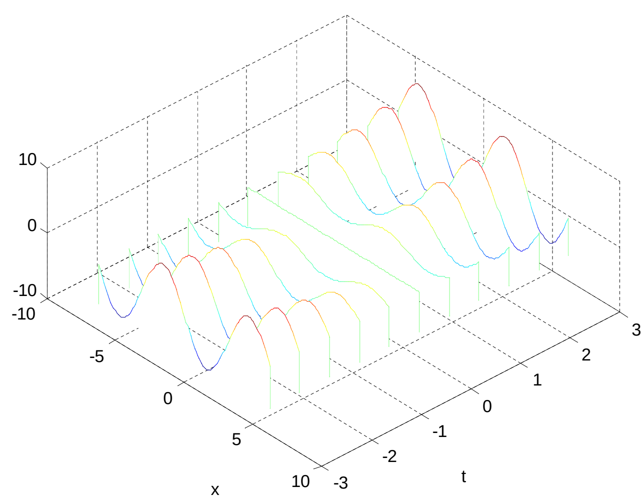
  (b). For a fix t, this is just a straight line with slope $\sin(2t)$. Also, the function is periodic in t. Therefore, as t moves on, you will see a family of straight lines that are swinging. See the pictures below:



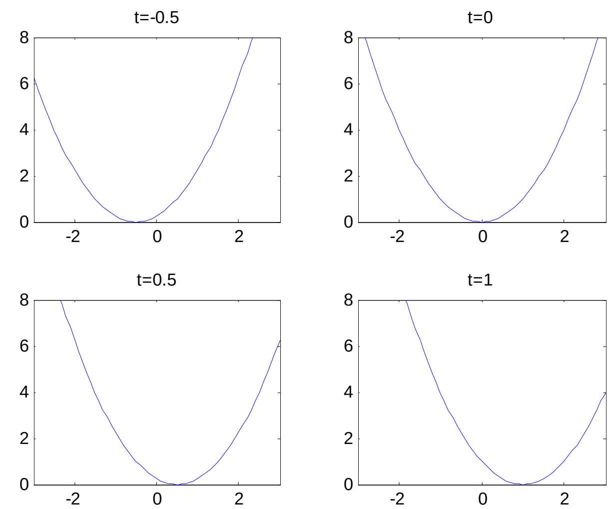


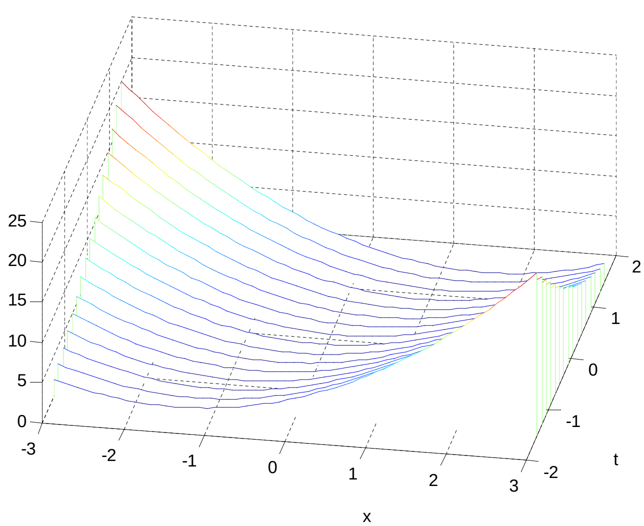
(d). Fixing t, it's just a sin function with amplitude $2|t|$. When t goes from negative infinity to 0, you'll see a decaying sin function which dampens to a straight line at t=0. After that you'll see a growing sin functions as t goes on. The crest for $t < 0$ will be the trough and vice versa. (You just switch the roles of x and t in (b) and you'll see this. In the above picture, if you look it the direction of x, you'll see the picture of this problem).



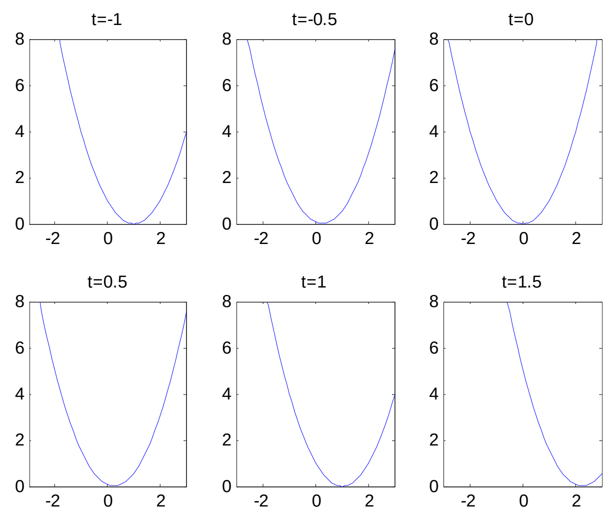


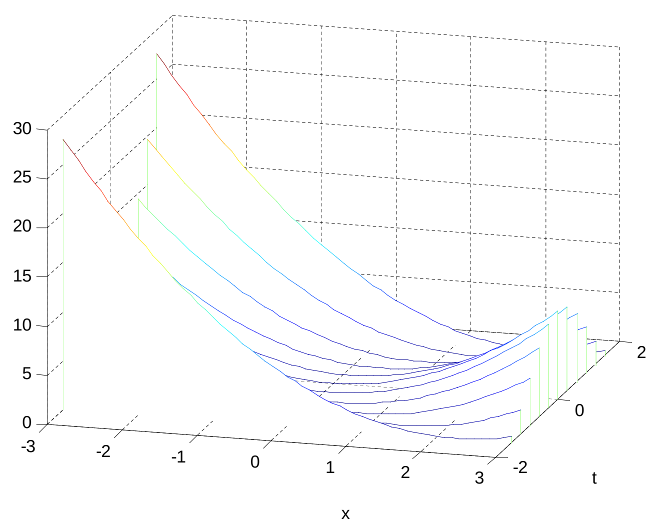
(f). The function is $f(x,t) = (x-t)^2$. For different t, it's just a parabola with the vertex at $(t, 0)$. Therefore, as t goes on, you'll see a right moving parabola with uniform speed 1.



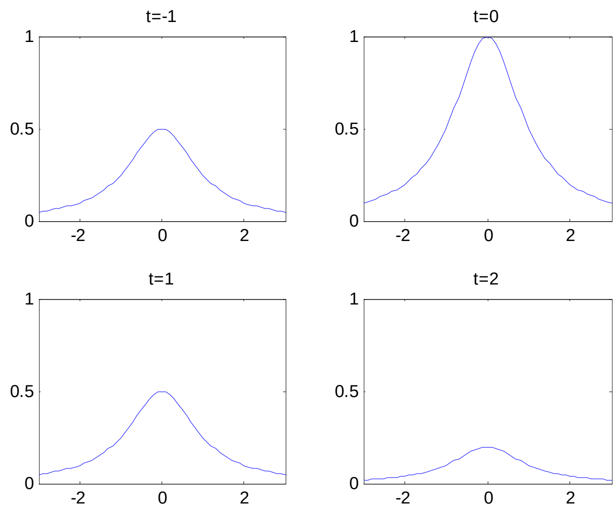


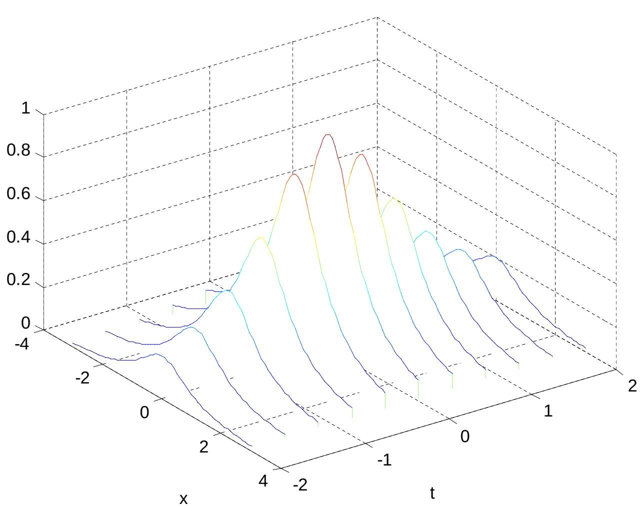
(h). The function is $f(x,t) = (x - t^2)^2$. For different t, it's a parabola with vertex at $(t^2, 0)$. You'll also see a left moving parabola when $t < 0$ with decreasing speed$(2|t|)$ and a right moving parabola with increasing speed $(2t)$.





(j)(Finally, the last 'movie'!) This function is the product of $\frac{1}{1+x^2}$ and $\frac{1}{1+t^2}$. Therefore, it's essentially Agnesi's witch but stretched. When t goes from negative infinity to 0, it grows higher and higher from 0 until $t = 0$. After that it dampens shorter and shorter.





- 16. To determine the direction of motion, we can focus on one specific value of the function $g(0)$. As time changes, this point will move so that we always have $x - ct = 0$. You can see that $x = ct$. That means this function value will be located at ct when time is t. Therefore, the wave speed is $x'(t) = c$. When $c > 0$, it's moving right and when $c < 0$, it's moving left.

  One example is $14(f)$. Look at the picture there and you'll see how the profile moves. In that case, $g(x) = x^2$ and $c = 1$.

Chapter 4.

1. (a). The function $f(x, y) = \frac{xy}{x^2+y^2}$ can't be made continuous at $(0, 0)$. Consider you are moving toward the origin on the line $y = kx$. The limit of this function on this line is:

$$\lim_{x \to 0} \frac{x * kx}{x^2 + k^2 x^2} = \frac{k}{1 + k^2}$$

which is different for different k values. Therefore, you can't find a limit if you approach the origin with an arbitrary fashion.

(b). This function can't be made continuous as well since it's easy to see that the function value approaches infinity as the point approaches the origin.

(c). This function can't be made continuous as well. We can choose two specific directions, for example the x axis and y axis.

When you approach on x axis, the limit is:

$$\lim_{y=0, x \to 0^+} h(x, y) = \lim_{x \to 0^+} \frac{x}{x^2} = +\infty$$
$$\lim_{y=0, x \to 0^-} h(x, y) = \lim_{x \to 0^-} \frac{x}{x^2} = -\infty$$

and when you approach on y axis, you have:

$$\lim_{x=0, y \to 0} h(x, y) = \lim_{y \to 0} 0 = 0$$

The limit doesn't exist.

(d). Similar reasoning as part (c). The limit doesn't exist. Can't be made continuous.

(e). This problem is a little bit tricky. The answer is yes now. You can try several direction and you'll always find 0 as the limit. Then, we may think the limit is really 0 no matter how you approach the origin. Then, we need to prove this.

Claim: $\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$

Proof. When $x = 0$, $q(x, y) = 0$ and when $x \neq 0$, we have:

$$0 \leq q(x,y) = \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2}{\sqrt{x^2}} = |x|$$

No matter which case, we always have(even when $x = 0$):

$$0 \leq q(x,y) \leq |x|$$

When $(x, y) \to (0, 0)$, both the left and the right go to 0. By sandwich theorem, $q(x, y)$ goes to 0. $\square$

Bingo!