An elementary proof of two formulas in trigonometry

By Lei, Sep. 7th

In the section today, I was asked why \( 1 + \cos(x) = 2\cos^2\left(\frac{x}{2}\right) \) and I wanted to prove \( \cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \). However I was stuck that time. After the section, I immediately realized it was actually very direct. Below, I’ll prove \( \cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b) \) and \( \sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b) \) for \( 0 \leq a, b \leq \frac{\pi}{2} \).

As for the general case, they are just some corollaries of these two basic equations.

A.

First of all, if \( a \) or \( b \) is equal 0 or \( \pi/2 \), the equations are obvious correct.

Now let’s look at the other cases.

Let’s look at the first one:

![Diagram of unit circle showing angles a and b and points O, A, B, C, D, E, and OB=cos(b) and thus OC=OB*cos(a)=cos(b)*cos(a);](image.png)
EC=BD=AB*\sin(\angle BAD)=AB*\sin(a)=\sin(b)\sin(a).

Hence OE=OC-EC, and this is actually \cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b).

Let's look at the second one.

We have AD=AB*\cos(BAD)=AB*\cos(a)=\sin(b)\cos(a) and DE=BC=OB*\sin(a)=\cos(b)\sin(a).

Hence \sin(a+b)=AE= DE+AD=\sin(a)\cos(b)+\cos(a)\sin(b).

B. For general \ a and b, we can use that \ \sin(x) = \sin (x + 2k\pi) , \ \cos(x) = \cos (x + 2k\pi) \ \cos(-x)=\cos(x) , \ \sin(\pi - x) = \sin (x) , \ \text{and} \ \cos(\pi - x) = -\cos (x) \ \text{etc to reduce them to the above cases. We can see that the two equations are also right.}

\begin{align*}
\cos(a+b) &= \cos(a)\cos(b)-\sin(a)\sin(b) \\
\sin(a+b) &= \sin(a)\cos(b)+\cos(a)\sin(b)
\end{align*}

C. By replacing b as \ -b, we have:

\begin{align*}
\cos(a-b) &= \cos(a)\cos(b)+\sin(a)\sin(b)
\end{align*}
\[ \sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b) \]

D. We can also have from the above equations that:

\[ \cos(a)\cos(b) = \frac{\cos(a+b) + \cos(a-b)}{2} \]

\[ \sin(a)\sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2} \]

\[ \sin(a)\cos(b) = \frac{\sin(a+b) + \sin(a-b)}{2} \]

\[ \cos(a)\sin(b) = \frac{\sin(a+b) - \sin(a-b)}{2} \]

E. Let \( a=b=x \), and we have:

\[
\begin{align*}
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \\
\sin(2x) &= 2\sin(x)\cos(x)
\end{align*}
\]