Keys to Sample

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1. $A(2,6,3)$ and $P_1 : -3x + 2y + z = 5$.

   For the distance, you need one normal vector of the plane P_1. Look at the equation for plane $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. You only need to pick the coefficients of the x, y, z to obtain a normal vector. We pick:

   $$\vec{n} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

   According to the formula, you need a point on the plane. This is easy, for example, we can choose $Q(0,0,5)$. You can pick any point that satisfies the equation. Then:

   $$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} =< 2,6,3 > - < 0,0,5 >=< 2,6,-2 >$$

   Then, the distance is:

   $$d = |\frac{\overrightarrow{QP} \cdot \vec{n}}{|\vec{n}|}| = |\frac{-6+12-2}{\sqrt{14}}| = \frac{2\sqrt{14}}{7}$$

   To find a sphere, you need the center (x_0, y_0, z_0) and radius r. The equation then is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

   We can have $r = d = \frac{4}{\sqrt{14}}$. Then the equation is

   $$(x-2)^2 + (y-6)^2 + (z-3)^2 = 8/7$$

2. We need to get $x'(t)$ and $y'(t)$. $x'(t) = e^t$ and $y'(t) = 3e^{3t/2}$. Then the length is:

   $$\begin{aligned} \int_0^1 \sqrt{(e^t)^2 + (3e^{3t/2})^2} dt &= \int_0^1 \sqrt{e^{2t} + 9e^{3t}} dt \\ &= \int_0^1 e^t \sqrt{1 + 9e^t} dt \end{aligned}$$

   Do you know how to do the following now? Just substitution $u = 9e^t + 1$.

   Ans: $\frac{2}{27}((1+9e)^{3/2} - \sqrt{10}^3)$

   For the slope, you only need to calculate $\frac{dy}{dx}|_{t=0} = \frac{y'(t)}{x'(t)}|_{t=0} = 3$

   For the nonparametric equation, you just need to eliminate t. First of all, notice the range of x and y: $x > 0, y > 0$. Then $y = 2(e^t)^{3/2} = 2x^{3/2}, x > 0$

   I'm sorry. I don't want to draw that picture here, even though programming in Matlab is easy.

3. A sequence converges means the sequence has a limit. A series converges means the $n-th$ partial sum has a limit.

   (a). You can see that as $n \to \infty$, the dominate term of the numerator is n. Dividing it, you'll get:

$$\frac{n}{2n+1} = \frac{1}{2+1/n} \to 1/2$$

   The sequence converges to 1/2.

   (b). By definition, the series also converges to 1/2.

   Attention: Suppose we have the series $\sum_{n=1}^{\infty} \frac{1}{n}$. The sequence of the $n-th$ term $1/n$ converges to 0, but the sequence of the $n-th$ partial sum diverges. Thus this series diverges. For this problem, we know the expression of the $n-th$ partial sum. I hope you can distinguish the difference.

   (c). Since the series converges, the $n-th$ term must converge to 0. Otherwise, it'll contradict with the $n-th$ term for divergence.

   (d). $a_1 = S_1 = 1/3$. $a_2 = S_2 - S_1 = 2/5 - 1/3 = 1/15$.

   Generally, for $k \geq 2$,

$$a_k = S_k - S_{k-1} = \frac{k}{2k+1} - \frac{k-1}{2k-1} = \frac{1}{4k^2-1}$$

   Check whether $k = 1$ satisfies this expression. You can see it does.

   Exercise: Do the same problems with $S_n = 2n^2 + 1$.

   Also do for $S_n = \ln(n+1)$

4. (a). Two methods. First of all, simplify the numerator first $1/2 + i\sqrt{3}/2 = r_1 e^{i\theta_1}$. $r_1 = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$. $\theta_1 = \pi/3$. Then we have $z = \frac{e^{i\pi/3}}{e^{-i\pi/3}} = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. You can also expand the denominator first and then multiply the conjugate of denominator.

   (b). It looks like a geometric series. Let's see. Write out the first several terms: $4, 16/5, 64/25, 4^4/5^3, \ldots$. You can see that we can get the latter term by multiplying 4/5 with the former term. Thus the multiplier is a constant and the series is geometric. $first = 4, multiplier = 4/5 < 1$. It converges and $sum = \frac{4}{1-4/5} = 20$.

   Exercise: What if k is from 5?

5. $r = 1 + 2\cos\theta$

(a). Since the symmetric point for (r, θ) about x-axis is $(r, -\theta + 2k\pi)$ or $(-r, \pi - \theta + 2k\pi)$. We need to check whether one of these expressions satisfies the equation provided (r, θ) satisfies the equation. Since $\cos\theta = \cos(-\theta)$, we can see that $(r, -\theta)$ works. Then, since $(r, -\theta)$ is on the curve, we get the conclusion.

(b). I don't want to draw here.

(c). Use the formula $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

You can figure out that $f'(\theta) = -2\sin\theta$. Then if $\theta = 2\pi/3$, we'll have $f(\frac{2\pi}{3}) = 0$ and $f'(\frac{2\pi}{3}) = -\sqrt{3}$. Plug in, and you'll discover:

$$\frac{dy}{dx}|_{\theta=2\pi/3} = \tan(2\pi/3) = -\sqrt{3}$$

(d). I believe all of you should know the formula:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

We need to determine the limits.

In the upper-left quadrant, $x \leq 0, y \geq 0$. Let's observe this curve. From $\theta = 0$ till $\theta = 2\pi/3$, $r > 0$. We can know that the curve enters this quadrant at $\theta = \pi/2$ and leaves it at $2\pi/3$. After this, even θ is less than π, the r value is negative and the curve isn't here. We get:

$$A = \int_{\pi/2}^{2\pi/3} \frac{1}{2}(1 + 2\cos\theta)^2 d\theta = \frac{1}{2}\int_{\pi/2}^{2\pi/3} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$

As we were tested in the quiz, we need to change $\cos^2\theta$ into $\frac{1+\cos(2\theta)}{2}$. Then the integral becomes:

$$A = \frac{1}{2}\int_{\pi/2}^{2\pi/3} (1 + 4\cos\theta + 2(1 + \cos(2\theta))) d\theta = \frac{1}{2}(3\theta + 4\sin\theta + \sin(2\theta))|_{\pi/2}^{2\pi/3}$$

$\sin(4\pi/3) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$ and $\sin(2\pi/3) = \frac{\sqrt{3}}{2}$.

Ans: $\frac{\pi}{4} + (\sqrt{3} - 2) - \frac{\sqrt{3}}{4} = \frac{\pi}{4} + \frac{3\sqrt{3}}{4} - 2$

6. (a). We can see that it's first order ODE. It's separable. Put all y's on the left hand side and all x on the right hand side. We have $\frac{dy}{\sqrt{y^3}} = \sqrt{x}dx$. Integrate, we have:

$$\int \frac{1}{\sqrt{y^3}} dy = \int \sqrt{x} dx$$

$$\frac{1}{1 - 3/2} y^{1-3/2} = \frac{2}{3} x^{3/2} + C$$

$$-2y^{-1/2} = \frac{2}{3} x^{3/2} + C$$

Plug in $x = 0, y = 4$, we have $-2(\frac{1}{2}) = C$. Thus $C = -1$. Finally, we have $-2y^{-1/2} = \frac{2}{3}x^{3/2} - 1$ or $y^{-1/2} = -\frac{1}{3}x^{3/2} + \frac{1}{2}$. $y(1) = 36$

(b). For this type, we can do trig substitution $x = 2\sin\theta$ (refer to the material for review). And then $dx = 2\cos\theta d\theta$. The integral thus becomes:

$$\int \frac{1}{(4 - 4\sin^2\theta)^{3/2}} 2\cos\theta d\theta = \int \frac{2\cos\theta}{8\cos^3\theta} d\theta = \frac{1}{4}\int \sec^2\theta d\theta = \frac{1}{4}\tan\theta + C$$

The variable is x, so you must express this result in terms of x. Draw the triangle. $\sin\theta = \frac{x}{2}$. You can get the other edge is of length $\sqrt{4 - x^2}$. Thus,$\tan\theta = \frac{x}{\sqrt{4-x^2}}$.
Ans: $\frac{x}{4\sqrt{4-x^2}} + C$

7. (a). You just need to notice that x is somewhat the derivative of $x^2 - 4$ (except the constant). You thus do the substitution $u = x^2 - 4$ and $du = 2xdx$. $x = 2, u = 0$ and $x = 3, u = 5$. The improper integral becomes:

$$\int_0^5 \frac{1}{u^{1/3}} \frac{1}{2} du = \frac{1}{2}\int_0^5 u^{-1/3} du = \frac{1}{2}\frac{3}{2} u^{2/3}|_0^5 = \frac{3}{4}\sqrt[3]{25}$$

(b). We can see that $\sqrt{x}$ looks unnatural, so we do substitution $u = \sqrt{x}$ and we have $du = \frac{1}{2\sqrt{x}}dx$. $dx = 2\sqrt{x}du = 2udu$.

$$\int \frac{u^2}{1+u} 2udu = \int \frac{2u^3}{u+1} du$$

Integration of a fraction. The fraction is improper. Long division. You'll get: $\frac{2u^3}{u+1} = 2u^2 - 2u + 2 - \frac{2}{u+1}$.

$$\frac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1| + C = \frac{2}{3}x\sqrt{x} - x + 2\sqrt{x} - 2\ln|\sqrt{x} + 1| + C$$

8. We need to get the Taylor polynomial for $f(x)$. However, we don't know which degree we should use. We can assume N at first. We can see that $\sqrt{3} = f(-1)$. This means $x = -1$. If we want to decimal places, we want $|error| < 0.01$.

$$\begin{aligned}
f(x) &= \sqrt{4+x} \\
f'(x) &= \frac{1}{2}(x+4)^{-1/2} \\
f''(x) &= -\frac{1}{2\cdot 2}(x+4)^{-3/2} \\
f'''(x) &= \frac{3}{2\cdot 2\cdot 2}(x+4)^{-5/2} \\
f^{(4)}(x) &= -\frac{3\cdot 5}{2\cdot 2\cdot 2\cdot 2}(x+4)^{-7/2} \\
&\cdots
\end{aligned}$$

At $a = 0$, we'll have $f(0) = 2, f'(0) = 1/4, f''(0) = -\frac{1}{2^5}, f^{(3)}(0) = \frac{3}{2^8}$ and ....

We have two methods of error estimation. Let's see whether both of them work here. Let's use the remainder term of Taylor polynomial first. We know the remainder term is:

$$\frac{f^{(N+1)(c)}}{(N+1)!}|x-a|^{N+1}$$

Here, $|x - a| = | - 1 - 0| = 1$ and we have $\frac{f^{(N+1)}(c)}{(N+1)!}$. Let's see which degree we can choose. According to what has been calculated, we may hope $f^{(N+1)(c)}/(N + 1)!$ be close to $f^{(N+1)}(0)/(N + 1)!$. Then, you'll see that $f'''(0)/3! < 0.01$. We thus hope that we can choose $N + 1 = 3$, namely $N = 2$. We of course should check:

$$\frac{f^{(3)}(c)}{3!} = \frac{3(c+4)^{-5/2}}{6*8}$$

Since c is between x and a, namely -1 and 0. Then $(c + 4)^{-5/2}$ is less than $3^{-5/2}$. Thus the remainder term is less than $\frac{3*3^{-5/2}}{6*8} = \frac{1}{3\sqrt{3}*48} < 0.01$. Then $T_2^0 f$ works.

We have $T_2^0 f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 2 + \frac{1}{4}x - \frac{x^2}{64}$. We use $T_2 f(-1) = 2 - 0.25 - 1/64 \approx 1.73$ to approximate $\sqrt{3}$.

Another method is limited to convergent alternating series. Here, you can notice that the derivatives are alternating after the second term. However, you can check that if $x = -1$, the Taylor series is not alternating since x is negative. The second method won't work.