Material for review. By Lei. May, 2011

You **shouldn't** only use this to do the review. Read your book and do the example problems. Do the problems in Midterms and homework once again to have a review.

Some suggestions:

1). Stay until last minute in the final.

2). Do easy problems first. If you can't finish one problem quickly or feel confused, just skip it first and come back later.

3). Check your answer using different ways. For example, differentiate back to check your integral. Plug in the solution to differential equations to check. Check $\vec{a} \times \vec{b}$ by taking the dot product of your answer with the two vectors to see whether they are zeros.

4).Memorize some important conclusions and theorems. For example, some basic integration formulas, integration by parts, trigonometric integrals/substitution, L'Hopital's rule, p-series, telescoping/geometric series. Taylor series, two methods of error estimation (Remainder term and AST). Euler's identity. Integrating factor, some forms of particular solutions to try. Formulas for slope, area and length. Transform between Cartesian and polar. Dot product, cross product, projection, area using vectors, distance between a point and a line or a plane, angle between planes, etc.

1 Integration

1.1 Basic Formulas and substitution

$$\int u^\alpha du = \frac{1}{\alpha+1}u^{\alpha+1} + C(\alpha \neq -1) \qquad \int \frac{1}{u} du = \ln|u| + C$$

$$\int \sec^2 u du = \tan u + C$$

$$\int \sec u \tan u du = \sec u + C \qquad \int \tan u du = \ln|\sec u| + C$$

$$\int a^u du = \frac{a^u}{\ln a} + C \qquad \int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}(u/a) + C$$

$$\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}(u/a) + C \qquad \int \sec u du = \ln|\sec u + \tan u| + C$$

Example: $\int \frac{1}{\sqrt{8x-x^2}}dx \quad \int e^{\tan u}\sec^2 u du \quad \int \frac{4u}{1+4u^2}du \quad \int y\sqrt{4y^2+1}dy$

1.2 Integration by parts and reduction formulas

$$\int udv = uv - \int vdu$$

$$\text{Example:} \int (x^2+2x)e^x dx \qquad \int (x^2+2x)\sin x dx \qquad \int \cos x e^x dx$$

$$I_n = \int x^n e^x dx \qquad I_n = \int x^n \sin x dx (\text{Or} \cos x) \quad I_n = \int x^n \ln x dx$$

$$I_n = \int \tan^n x dx \quad I_n = \int \frac{1}{(1+x^2)^n}dx \quad I_n = \int \sin^n x dx \quad \text{etc}$$

1.3 Integration of partial fractions

I won't list the formulas. Just remind you two things:

1). Fraction must be polynomial over polynomial. If not, you can try whether you can get polynomial over polynomial by substitution or something else.

2). Check whether it's improper fraction first. If it's improper, reduce it first.

Example: $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx \quad \int \frac{1}{x(x^2+1)^2}dx \quad \int \frac{x^2\sin^2 x - \cos^2 x + x}{x^2+1}dx$
$\int \frac{e^x}{e^{2x}-5e^x+6}dx \quad \int \frac{1}{(\sqrt{x}+2)(\sqrt{x}-3)}dx$

Note: Cover-up only applies for linear factors. Multiply and then take special x's to get coefficients may be fast sometimes.

1.4 Trig integrals and trig substitution

Example: $\int \sin^3 x \cos^2 x dx \quad \int \sec^3 x dx$ or $\int \tan^2 x \sec x dx$

$\sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Example: $\int \sqrt{1+\cos 4x}dx$

$$\sqrt{a^2+x^2} \quad x = a\tan\theta \qquad \sqrt{1+a^2x^2} \quad ax = \tan\theta$$
$$\sqrt{a^2-x^2} \quad x = a\cos\theta \ or \ a\sin\theta \qquad \sqrt{x^2-a^2} \quad x = a\sec\theta$$

Example: $\int_0^2 \sqrt{4-x^2}dx \quad \int \sqrt{1+x^2}dx \quad \int \frac{1}{x^2\sqrt{4-x^2}}dx$

1.5 Improper Integrals

We only learned definition and comparison test.

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$
$$\int_a^b f(x)dx = \lim_{c\to a}\int_c^b f(x)dx \quad \text{if a is a singular point}$$

Example: $\int_0^\infty x^2e^{-x}dx$ $\int_0^1 \frac{1}{\sqrt{x}}dx$ When does $\int_0^\infty \frac{1}{x^k}dx$ converge?

Tests for convergence:

Definition; Direct comparison; Limit Comparison(We didn't learn this, but this may be helpful sometimes)

Example: $\int_0^\infty \frac{1}{e^x+5}dx$ converges? $\int_2^\infty \frac{1}{x(\ln x)^p}dx$ $\int_2^\infty \frac{1}{2x^3-1}dx$

For this kind of problems, you can usually take the dominant terms to see.

2 Series

2.1 Sequences

(We didn't learn much)

Just some limits. Sandwich, L'Hopital and some other techniques may be used.

Several important conclusions.

$$(\ln n)^p \ll n^q \ll a^n \ll n! \ll n^n \quad p>0, q>0, a>1$$
$$\sqrt[n]{n}\to 1 \quad \sqrt[n]{x}\to 1, x>0 \quad x^n\to 0\ |x|<1 \quad (1+\frac{x}{n})^n\to e^x, \quad n\to\infty$$

2.2 Three important kinds of series

Geometric, p-series, telescoping

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{first}{1-ratio} = \frac{a}{1-r}, |r|<1$$
$$\text{Diverges if } |r|\geq 1$$

Example: $\sum_{n=5}^\infty \frac{\cos(n\pi)7^{2n}}{100^n}$ $\sum_1^\infty \frac{1}{n(n+1)}$
$\sum_1^\infty \frac{1}{n^{1.1}}$ $\sum_1^\infty(\tan^{-1}(n+1)-\tan^{-1}(n))$

2.3 Tests for convergence

Integral test, Direct Comparison Test, AST, Absolute convergence test. We didn't learn Limit Comparison Test, Ratio Test, Root Test

We can follow these steps:

1. Check $n-th$ term for **divergence**
2. See if it's geometric, telescoping. If it is, calculate their sums or determine whether they are divergent.
3. See if it's p-series or if it can compare to geometric, p-series.
4. See if Integral test applies.
5. If the sign changes alternatively, check whether it converges absolutely and check whether AST applies.

Example: $\sum_1^\infty \frac{1}{n(1+\ln^2 n)}$ $\sum \cos(1/n)$ $\sum(-1)^n \frac{n+3}{n+2}$ $\sum(-1)^n \frac{1}{\sqrt{n}}$ Converge conditionally or absolutely?

Comparison test, we can usually take the dominant term to find the comparison series.

Example: $\sum \frac{\sqrt{n}}{n^2+1}$, $\sum \frac{1}{n\sqrt[n]{n}}$, $\sum \frac{2^n}{n^2}$ $\quad \sum(\frac{1}{n+1})^n$

2.4 Taylor and Maclaurin series

Two methods, one way is to use formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

The other way is to use some known expansions or integrate or differentiate known expansions.

Important expansions:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + \sum_{k=1}^{\infty} x^k \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\ln(1+x) = \int \frac{1}{1+x} dx = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

Example: 2^x, $x^2 \sin(x^3)$, $\int \tan^{-1}(x)dx$, $\frac{2+2x}{1-x}$, $\frac{1}{(1-x)(2-x)}$

What's $1 + 1 + \frac{1}{2} + \ldots + \frac{1}{n!} + \ldots$?

2.5 Taylor polynomial and two kinds of error estimation

One way is to use Taylor theorem. Only for alternating can we use AST. $f(x) = T_N^a f(x) + R_N(x)$:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!} + R_N(x), \quad R_N(x) = \frac{f^{(N+1)}(c)(x-a)^{N+1}}{(N+1)!}$$

Example:$x = 0.1$ $\sin x = x - x^3/3! + error$; $f(x) = \sqrt{1+x} = 1 + \frac{1}{2}x + error$ If $|error| < 0.1$, range of x?

3 Complex Numbers

Multiply the conjugate of the denominator to simplify.
Euler's identity $e^{ix} = \cos x + i \sin x$
Transform between $a + bi$ and $re^{i\theta}$. Draw Argand Diagram.
De Moivre's theorem (Using $e^{in\theta} = (e^{i\theta})^n$):

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

The nth roots of $z = a + bi = re^{i\theta}$ are $\sqrt[n]{r}\exp(i(\frac{\theta}{n} + k\frac{2\pi}{n}))$ $0 \le k \le n-1$.

Example: Simplify $z = \frac{\sqrt{2}+\sqrt{2}i}{1-\sqrt{3}i}$, express it as $re^{i\theta}$ and draw $\bar{z}$.
Solve $z^4 - z = 1$ in $\mathbb{C}$ and $\mathbb{R}$. Express $\sin(2\theta)$ and $\cos(2\theta)$ using $\sin\theta$ and $\cos\theta$
Can e^x be negative if x is real? How about complex? Can $\sin x$ be larger than 1 if x is complex?

4 Differential equations

4.1 First order

separable

$$\frac{dy}{dx} = G(x)H(y)$$

Put all y's on left hand side and all x's on the right hand side, and integrate.

Example: $2dx + y^2dx = dy/(x^3 + e^x)$ $\quad y' + y^2 = 4$ $\quad (1-t^2)\frac{dz}{dt} - 1 - z^2 = 0$

Linear

$$a(x)y'(x) + b(x)y = f(x)$$

First of all, reduce it to standard form $y'(x) + p(x)y = q(x)$ and then find the integrating factor

$$\mu(x) = e^{\int p(x)dx}$$

Example: $3xy' - y = \ln x + 1 \quad xy' + 3y = \sin x/x^2$

Or solve it by $y = y_h + y_p$.

4.2 Second order

We only talk about linear equations:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = G(x)$$

Homogeneous with constant coefficients

$$ay'' + by' + cy = 0$$

The corresponding auxiliary equation $ar^2 + br + c = 0$. Three cases.

Inhomogeneous

General solution is $y_c + y_p$. Using boundary conditions to determine constants

For constant coefficients, several cases for particular solutions:

RHS is e^{rx}. Try Ae^{rx} if r is not a root, Axe^{rx} if r is a single root and Ax^2e^{rx} if it is a double root.

RHS is polynomial. Try polynomial with suitable degree.

RHS is $\sin(rx)$ or $\cos(rx)$ or the combination of them. Try $A \sin rx + B \cos rx$ if $\sin rx, \cos rx$ are not solutions to the complementary equation, $Ax \sin rx + Bx \cos rx$ if $\sin rx, \cos rx$ are solutions to complementary equation.

Example: $y'' - y = e^x, y(0) = 1, y'(0) = 0 \; y'' - 4y' + 4y = e^x + e^{2x}$

Variation of parameters

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= G/a \end{aligned}$$

Example: $x^2y'' - xy' - 8y = x^2$ Given solution x^n. Notice $a = x^2$ if variation of parameters. You can also try particular form.

Application: Spring

$my'' + \delta y' + ky = f(t) \; \omega = \sqrt{k/m}$ and period $T = 2\pi/\omega$.

Solve. Sketch graph. See if it's damped or not. Overdamping, critical damping and underdamping.

5 Parametric equations+polar coordinates

5.1 Parametric curves

$x = x(t), y = y(t)$.

Draw the graph: Find many enough t values to get the pairs (x, y). Draw these points and connect them to get the rough picture. Maybe eliminate t to get the Cartesian equation and draw the accurate picture. Pay attention to the ranges of x and y. If two curves have the same expression but the ranges are different, they are different curves! (e.g. line segment vs line)

Example: Draw the graph of $x = t^2, y = t^3, t \in \mathbb{R}$ and indicate the direction of motion. Are $x = t, y = t^2$ $x = e^t, y = e^{2t}$ the same?

Some special curves: cycloid, circle and ellipse. What is $x = \cos t, y = \sin t$? What's the direction of motion? How about $x = \sin t, y = \cos t$? How about $x = \cos(-t), y = \sin(-t)$? Notice: You can calculate the velocity vector to get the direction of motion.

Slope: $y(t) = y(x(t))$. Differentiate with respect to t, and you'll get the slope:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example: $x^3 + 2t^2 = 9, 2y^3 - 3t^2 = 4$ at $t = 2$. Find tangent line of $x = 4\sin t, y = 2\cos t$ at $t = \pi/4$ and $t = \pi/2$.

Area bounded by $x = x(t_1)$, $x = x(t_2)$ and the curve. (You must compare to the area formula for polar coordinate):

$$A = \int_{t_1}^{t_2} y(t)x'(t)dt$$

Example: Find the area of ellipse $x = 2\cos t, y = \sin t$ in the $1st$ quadrant.

Length

$$L = \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}dt$$

Example: Find the length of $x = \sin t, y = t - \cos t$ for $0 \leq t \leq \pi$.

Parameterize the circle $x = 2\cos t, y = 2\sin t$ with arc length parameter and check that $\int_2^5 \sqrt{x'(s)^2 + y'(s)^2}ds = 5 - 2 = 3$ directly.

5.2 Polar coordinates

Plot the point $(-1, \pi/4)$ and find all the polar coordinates for it.

Relationship with Cartesian coordinate:

$$x = r\cos\theta, y = r\sin\theta$$
$$r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$$

Example: What is $r = 2\cos\theta$? Find the polar equation of $x^2 + xy + y^2 = 1$

Polar graphs:

Symmetry: Suppose (r, θ) is on the curve.
The point which is symmetric to it about x-axis has coordinates $(r, -\theta + 2k\pi)$ and $(-r, \pi - \theta + 2k\pi)$. If one of these expression also satisfies the equation, then the curve is symmetric about x-axis.
How about y? The coordinates are $(r, \pi - \theta + 2k\pi)$ and $(-r, -\theta + 2k\pi)$.
About the origin? $(-r, \theta + 2k\pi)$ and $(r, \pi + \theta + 2k\pi)$.

Curve $r = f(\theta)$:

Slope. Just regard θ as parameter and use the formula for parametric curves. $x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$. Then:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Example: Sketch $r = 1 - \cos\theta$. Identify symmetries and calculate the slope at $\theta = \pi/3$.
Note: If you don't know anything about the graph, you can calculate many enough pairs (r, θ) and plot them.

Area bounded by $\theta = \alpha$, $\theta = \beta$ and the curve:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

Example: Find the area of one leaf of the three-leaved rose $r = \cos(3\theta)$.
Find the area bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$. Also, find the area bounded by $x = x(\pi/4)$, $x = x(\pi/2)$ and the curve.

Length Just use the formula for parametric curves.

$$L = \int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

Find the length of spiral $r = \theta^2$ from the origin to where it meets positive x for the first time.

6 Vectors

6.1 Coordinate frame for space(3D)

Right handed. Cylinder, sphere, disk etc.

6.2 addition, subtraction, scalar multiplication

Example: Does $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$ make sense? How about $(\vec{u} \cdot \vec{v})\vec{w}$? How about $\vec{a}/|\vec{b}|$, $\vec{a}/\vec{b}$?

6.3 Length and direction, dot product, angle and projection

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ and $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ Proj_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{|v|^2} \vec{v} \\ \cos\theta &= \frac{\vec{u} \cdot \vec{v}}{|u||v|} \end{aligned}$$

Example: $P(-3,4,1), Q(-5,2,2), R(0,0,1)$ Find the length of $\overrightarrow{PQ}$, the cosine value of angle between $\overrightarrow{PQ}$ and $\overrightarrow{RP}$ and the cosine value of angle $\angle RPQ$
Example: $u = 6i + 3j + 2k, v = i - 2j - 2k$. Write u as components, one is perpendicular to v and one is parallel to v. Find angle between them.
Find the point C on the line AB where $A(1,2,1), B(-1,2,3)$ such that the distance between C and $D(0,1,0)$ is the smallest.
Knowing $|\vec{a}| = 3, |\vec{b}| = 7, \vec{a} \cdot \vec{b} = -2$, find $|\vec{a} - \vec{b}|$
(**) In parallelepiped $ABCD - A'B'C'D'$, if $|AA'| = 2, |AB| = 3, |AD| = 4, \angle BAD = \pi/3, \angle BAA' = \pi/6, \angle DAA' = \pi/4$, find the length of AC'.

6.4 Cross product, triple scalar product, area of triangles,parallelogram, volume

$\overrightarrow{PQ}\times\overrightarrow{PR} = |PQ|*|PR|*\sin\theta*\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both of them and satisfies the right hand principle. The area of the triangle should be $\frac{1}{2}|PQ|*|PR|\sin\theta$. Thus:

$$A = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}|$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

In plane or 2D, the area becomes:

$$Area = \frac{1}{2}\text{abs}\left(\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right)$$

Where $\vec{u} =< u_1, u_2 >= \overrightarrow{PQ}$ and $\vec{v} =< v_1, v_2 >= \overrightarrow{PR}$

How about the area of the parallelogram? $A(parallelegram) = |a\times b|$

Triple scalar product:

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{1}$$

Attention: No i, j, k here! Volume=|triple scalar|

Example: Find a unit vector orthogonal to PQR where $P(1,1,1)$, $Q(2,1,3)$, $R(3,-1,1)$. Find the area of this triangle and the area of the parallelogram determined by these three points.How about the points are $P(1,2), Q(2,1), R(3,-1)$?

Find the volume of parallelepiped determined by $\vec{u} = 2\hat{i}+\hat{j}, \vec{v} = 2\hat{i}-\hat{j}+\hat{k}, \vec{w} = \hat{i}+2\hat{k}$

What's the value of $(\vec{u}\times\vec{v})\cdot\vec{v}$? Is this true: $(\vec{u}\times\vec{v})\cdot\vec{w} = \vec{w}\cdot(\vec{v}\times\vec{u})$? $(\vec{a}+\vec{b})\times(\vec{a}+\vec{b}) =$? and $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) =$? ($\vec{0}$ and $2\vec{a}\times\vec{b}$)

Check orthogonality by dot product and parallelism by cross product or see if one is the multiple of another

Example: $\vec{u} = 2i - j + k, \vec{v} = -4i + 2j - 2k, \vec{w} = 2j + 2k$. Check the relationship between them.

6.5 Equations of lines, Planes

Both need a point on it and a vector.The line needs a vector parallel to it and the plane need a vector perpendicular to it. (Not necessarily unit)

$$P(x_0, y_0, z_0), \vec{v} =< a, b, c >, \vec{n} =< A, B, C >$$
$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

6.6 Distance between a point and a line, distance between a point and a plane, distance between two planes or a line and a plane

$$Dis(P, line) = |\frac{\overrightarrow{PS} \times \vec{v}}{|v|}|$$
$$Dis(P, plane) = |\frac{\overrightarrow{PS} \cdot \vec{n}}{|n|}|$$

The other two kinds of distances can be reduced to the distance between a point and a plane.

Example: Homework in 12.5. Find the distance between $P(2, -3, 4)$ and $x+2y+2z = 13$, the distance between P and the line through $Q(0, 0, 0)$ and perpendicular to this plane.

6.7 Angle between two planes, line where two planes intersect

The angle (always the acute one) between two planes is (Notice $\vec{n}_1, \vec{n}_2$ don't have to be unit):

$$\theta = \cos^{-1}(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|})$$

For the line where two planes intersect, since the line is on both planes, it should be perpendicular to both normal vectors and thus be parallel to $n_1 \times n_2$, which can be regarded as the vector parallel to it. Finding a point on the line is finding a (only one is enough) solution to both equations of the two planes.