Math222 Exercise Quick answers

1. Complex Number
A. suppose $z = \frac{2e^{i\pi/3}}{1 - \sqrt{3}i} = a + bi = re^{i\theta}$. Find $a, b, r = |z|, \theta = \arg(z)$

B. Solve the equation $z^4 - z = 0$ in $C$

C. Use De’Moivre's theorem to express $\sin(2\theta)$ using $\sin \theta, \cos \theta$

D. Draw the Graph of $\omega = \sqrt{2 + \sqrt{2}i}$. Don’t calculate $\omega^1/\omega$ and $i\omega$ and draw the pictures of them.

Ans: 1). First approach: $2e^{i\pi/3} = 2(\cos(\pi/3) + i \sin(\pi/3)) = 1 + \sqrt{3}i$.
Then $z = \frac{1+i\sqrt{3}}{1-\sqrt{3}i}$. Then multiply the conjugate of the denominator, we get
$z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. $r = \sqrt{a^2 + b^2} = 1$ and $\theta = 2\pi/3$.
Second approach, $1 - \sqrt{3}i = 2e^{-i\pi/3}$ and then the ans is $\frac{2e^{i\pi/3}}{2e^{-i\pi/3}}$

2). $(z^3 - 1) = 0$. Then, one root is 0 and the other three satisfy $z^3 = 1$ and $1 \cdot e^{i(0+2k\pi)/3} (0 \leq k \leq 2)$

3). The theorem says $\cos(2\theta) + i \sin(2\theta) = (\cos \theta + i \sin \theta)^2 = (\cos^2 \theta - \sin^2 \theta + i2 \sin \theta \cos \theta)$. Compare the imaginary parts $\sin(2\theta) = 2 \cos \theta \sin \theta$

4). For this number, $r = \sqrt{2 + 2} = 2$. Then $2 \cos \theta = \sqrt{2}$. $\omega = 2e^{i\pi/4}$.
$1/\omega = (1/2)e^{-i\pi/4}$. $i\omega$ just rotating the graph of $\omega$ by $90^\circ$

2. First Order ODE

2.1. Separable(Think about the form of separable equations)
$\frac{dy}{dx} - xy^2 = 9x$
$y' = ky$

Note: Are they linear?

Ans: For the first, we have $\frac{dy}{dx} = x(9 + y^2)$ and then $\int \frac{1}{9+y^2}dy = \int xdx$.
$\frac{1}{3}\tan^{-1}(y/3) = x^2/2 + C$
$dy/y = kdx$ and then $|y| = Ae^{kx}$. $y = Be^{kx}$

For linear first order equation, see below. The first is not linear and the second is both separable and linear.

2.2. Linear(Think about the definition of linear equations.)
$y' + (\tan x)y = \sec x (\text{Integrating factor})$
$y' + 4y = 5 (\text{Using integrating factor and undetermined coefficients to solve).}$

What’s the limit $\lim_{x \to +\infty} x^3(y_1(x) + 5/4)$ where $y_1(0) = 1$.

Ans: Linear first order is like: $a(x)y' + b(x)y = f(x)$
The first equation can be solve by integrating factor. Notice that it is linear, and $\mu(x) = e^{\int \tan x dx}$, pick $\mu(x) = \sec x$. Then, multiply this on both sides,
((sec \(x\))y)' = sec^2 x \text{ and then } (sec \(x\))y = \tan x + C \text{ and } y = \sin x + C \cos x

For the second, the \(y' + 4y = 0\) \(y = Ce^{-4x}\) (a typo again...). Find the particular solution \(-5/4\). Then \(y(x) = Ce^{-4x} - 5/4\). Then \(x^3e^{-4x}\), using L'H, the limit is 0.

3. Second Order
3.1. General theory
(From Quiz 7) If I tell you that two solutions to the equation \(x^2y'' - 5xy' + 9y = 0\) are of the type \(y_1 = x^r\) and \(y_2 = x^r\ln x\) (here, the two \(r\)'s are the same), which are obviously linearly independent, find \(r\) and write out the general solution.

What if the right hand side is \(9\)? \(9 + x^2\) (\(x^3\) is too hard for you, use \(x^2\) here)?

ans: For the first part, please see the answer to the quiz. For the second part, the right hand side is not zero, then it’s inhomogeneous. Since, we find \(y_c = C_1x^3 + C_2x^3\ln x\), we only need to find \(y_p\). For \(9\), \(y_{p1} = 1\). For \(x^2\), try \(Ax^2\). Then, \(y = y_c + y_p\). The answer for first is \(y(x) = C_1x^3 + C_2x^3\ln x + 1 + x^2\)

3.2. 2nd linear ODE with constant coefficients
\(y'' + 4y = \sin(2x)\)

\(y'' + 2y' - 3y = x^2 + e^x. y(0) = 0, y'(0) = 0\)

\(y'' - 4y' + 4y = e^{2x}\)

ans: The homogeneous solution, \(r^2 + 4 = 0\) and then \(r^2 = -4, r = \pm 2i\).
\(y_c(x) = C_1\cos(2x) + C_2\sin(2x)\). For the particular solution, notice that \(\sin(2x)\) is a part of the general solution, try \(A_1x\cos(2x) + A_2x\sin(2x)\).
\(y_p = -\frac{1}{4}x\cos(2x)\).

(a typo here) \(r^2 + 2r - 3 = 0, r = -3, 1\). \(y_c = C_1e^{-3x} + C_2e^x\). For \(x^2\), just try \(Ax^2 + Bx + C\) since the coefficient of \(y\) is nonzero. For \(e^x\), since it’s a single root, try \(Dxe^x\).

\(r^2 - 4r + 4 = 0, r = 2\) is a double root. \(y_c = C_1e^{2x} + C_2xe^{2x}\). For particular solution, since \(e^{2x}, xe^{2x}\) are both solutions to the homogeneous equation, we must try \(y_p = Ax^2e^{2x}\).

3.3. Spring-Mass Discuss your homework.
The formula is \(my''(t) + \delta y'(t) + ky(t) = f(t)\). Notice the positive direction!
An undamped mass-spring system with a mass of 1kg, has a period of 1s.
What is the spring constant \(k\)? If the initial condition is \(y(0) = 1m\), and \(y'(0) = 1m/s\), find the motion of the mass. Make a careful graph of the
motion $y(t)$.

Since, it’s undamped, $\delta = 0$. We don’t mention the force, so $f(t) = 0$. Then, $1 \ast y'' + ky = 0$. It has a period of 1. Solve this equation, $C_1 \cos(\sqrt{k}t) + C_2 \sin(\sqrt{k}t)$, we can find the period is $2\pi/\sqrt{k}$. (The general formula is $T = 2\pi\sqrt{m/k}$). $k = 4\pi^2$. Then, $y(t) = C_1 \cos(2\pi t) + C_2 \sin(2\pi t)$. Then, determine the constants. Here, upward is positive and $y(0) = 1$ and $y'(0) = 1$, you can determine the coefficients.