Math222 Exercise Quick answers

1. Complex Number
A. suppose $z = \frac{2e^{i\pi/3}}{1-\sqrt{3}i} = a + bi = re^{i\theta}$. Find $a, b, r = |z|, \theta = arg(z)$
B. Solve the equation $z^4 - z = 0$ in C
C. Use De'Moirve's theorem to express $\sin(2\theta)$ using $\sin\theta, \cos\theta$
D. Draw the Graph of $\omega = \sqrt{2} + \sqrt{2}i$. Don't calculate $\overline{\omega}$ $1/\omega$ $i\omega$ and draw the pictures of them.
Ans: 1). First approach: $2e^{i\pi/3} = 2(\cos(\pi/3) + i\sin(\pi/3)) = 1 + \sqrt{3}i$. Then $z = \frac{1+i\sqrt{3}}{1-\sqrt{3}i}$. Then multiply the conjugate of the denominator, we get $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. $r = \sqrt{a^2 + b^2} = 1$ and $\theta = 2\pi/3$.
Second approach, $1 - \sqrt{3}i = 2e^{-\pi/3}$ and then the ans is $\frac{2e^{i\pi/3}}{2e^{-i\pi/3}}$
2). $z(z^3 - 1) = 0$. Then, one root is 0 and the other three satisfy $z^3 = 1$ and $1 * e^{i(0+2k\pi)/3}$ $(0 \leq k \leq 2)$
3). The theorem says $\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta)$. Compare the imaginary parts $\sin(2\theta) = 2\cos\theta\sin\theta$
4). For this number, $r = \sqrt{2+2} = 2$. Then $2 * \cos\theta = \sqrt{2}$. $\omega = 2e^{i\pi/4}$. $1/\omega = (1/2)e^{-i\pi/4}$. $i\omega$ just rotating the graph of ω by $90°$

2. First Order ODE
2.1. Separable(Think about the form of separable equations)
$\frac{dy}{dx} - xy^2 = 9x$
$y' = ky$
Note: Are they linear?
Ans: For the first, we have $\frac{dy}{dx} = x(9 + y^2)$ and then $\int \frac{1}{9+y^2}dy = \int xdx$. $\frac{1}{3}\tan^{-1}(y/3) = x^2/2 + C$
$dy/y = kdx$ and then $|y| = Ae^{kx}$. $y = Be^{kx}$
For linear first order equation, see below. The first is not linear and the second is both separable and linear.

2.2. Linear(Think about the definition of linear equations.)
$y' + (\tan x)y = \sec x$(Integrating factor)
$y' + 4y = 5$ (Using integrating factor and undetermined coefficients to solve). What's the limit $\lim_{x\to+\infty} x^3(y_1(x) + 5/4)$ where $y_1(0) = 1$.
Ans: Linear first order is like: $a(x)y' + b(x)y = f(x)$
The first equation can be solve by integrating factor. Notice that it is linear, and $\mu(x) = e^{\int \tan x dx}$, pick $\mu(x) = \sec x$. Then, multiply this on both sides,

$((\sec x)y)' = \sec^2 x$ and then $(\sec x)y = \tan x + C$ and $y = \sin x + C\cos x$
For the second, the $y' + 4y = 0$ $y = Ce^{-4x}$ (a typo again....). Find the particular solution $-5/4$. Then $y(x) = Ce^{-4x} - 5/4$. Then x^3e^{-4x}, using L'H, the limit is 0.

3. Second Order
3.1. General theory
(From Quiz 7) If I tell you that two solutions to the equation $x^2y''-5xy'+9y = 0$ are of the type $y_1 = x^r$ and $y_2 = x^r \ln x$ (here, the two r's are the same), which are obviously linearly independent, find r and write out the general solution.
What if the right hand side is 9? $9+x^2(x^3$ is too hard for you, use x^2 here)?
ans: For the first part, please see the answer to the quiz. For the second part, the right hand side is not zero, then it's inhomogeneous. Since, we find $y_c = C_1x^3 + C_2x^3 \ln x$, we only need to find y_p. For 9, $y_{p1} = 1$. For x^2, try Ax^2. Then, $y = y_c + y_p$. The answer for first is $y(x) = C_1x^3 + C_2x^3 \ln x + 1$ and the second is $y(x) = C_1x^3 + C_2x^3 \ln x + 1 + x^2$

3.2. 2nd linear ODE with constant coefficients
$y'' + 4y = \sin(2x)$
$y'' + 2y' - 3y = x^2 + e^x.y(0) = 0, y'(0) = 0$
$y'' - 4y' + 4y = e^{2x}$
ans: The homogeneous solution, $r^2 + 4 = 0$ and then $r^2 = -4, r = \pm 2i$. $y_c(x) = C_1\cos(2x) + C_2\sin(2x)$. For the particular solution, notice that $\sin(2x)$ is a part of the general solution, try $A_1x\cos(2x) + A_2x\sin(2x)$. $y_p = -\frac{1}{4}x\cos(2x)$.
(a typo here) $r^2 + 2r - 3 = 0$, $r = -3, 1$. $y_c = C_1e^{-3x} + C_2e^x$. for x^2, just try $Ax^2 + Bx + C$ since the coefficient of y is nonzero. For e^x, since it's a single root, try Dxe^x.
$r^2 - 4r + 4 = 0$. $r = 2$ is a double root. $y_c = C_1e^{2x} + C_2xe^{2x}$. For particular solution, since e^{2x}, xe^{2x} are both solutions to the homogeneous equation, we must try $y_p = Ax^2e^{2x}$.

3.3. Spring-Mass Discuss your homework.
The formula is $my''(t) + \delta y'(t) + ky(t) = f(t)$. Notice the positive direction! An undamped mass-spring system with a mass of $1kg$, has a period of $1s$. What is the spring constant k? If the initial condition is $y(0) = 1m$, and $y'(0) = 1m/s$, find the motion of the mass. Make a careful graph of the

motion y(t).
Since, it's undamped, $\delta = 0$. We don't mention the force, so $f(t) = 0$. Then, $1 * y'' + ky = 0$. It has a period of 1. Solve this equation, $C_1 \cos(\sqrt{k}t) + C_2 \sin(\sqrt{k}t)$, we can find the period is $2\pi/\sqrt{k}$. (The general formula is $T = 2\pi\sqrt{m/k}$). $k = 4\pi^2$. Then, $y(t) = C_1 \cos(2\pi t) + C_2 \sin(2\pi t)$. Then, determine the constants. Here, upward is positive and $y(0) = 1$ and $y'(0) = 1$, you can determine the coefficients.